LECTURE 3: APPLICATION TO COVER TIMES - BOUNDS

We discuss an application of the generalized Ray-Knight theorem to the study of cover times. In the setup of Lecture 1, we define the cover time as the first time that all vertices have been covered:

$$t_{\rm COV} = \inf\{t : \ell_y(t) > 0 \,\text{for all}\, y \in V\}.$$

A-priori, the cover time does depend on the starting point.

We first prove the Matthew bound, an a-priori upper bound on the cover time. We next show how the generalized second Ray-Knight theorem gives directly upper bounds on the cover time. Finally, we discuss an extension of the generalized second Ray-Knight theorem to a metric extension of V, which yields complimentary lower bounds.

A further discussion of the relation between cover time and Gaussian fields will hopefully be given in a subsequent lecture.

1. The Matthew bound

We discuss in this section the discrete random walk on G. We will not use the fact that the chain is reversible, so in this section we simply declare $\{X_n\}_{n\geq 0}$ to be a Markov chain with finite state space S. We set

$$T_{\rm COV} = \min\{n : \{X_0, \ldots, X_n\} = S\}.$$

Let $\tau_x = \min\{n : X_n = x\}$ denote the hitting time of x. Let $H(x, y) = E^x(\tau_y)$ denote the expected hitting time of y when starting at x. Let $h_{\max} = \max_{i,j} H(i, j)$.

Theorem 1.1 (Matthew)**.**

$$\max_{x\in S} E^x T_{\rm COV} \leq h_{\max} \cdot \sum_{k=1}^{|S|} \frac{1}{k}. \tag{1}$$

Proof. It is convenient to identify S with $\{1, \ldots, N\}$. Let σ denote a random permutation of S. Define $\mathcal{B}_i = \{\tau_{\sigma(i)} > \max_{j<i} \tau_{\sigma(j)}\}$, i.e. $\sigma(i)$ is visited last among $\sigma(1), \ldots, \sigma(j)$. Note that $\mathcal{B}_i$ is measurable on $\mathcal{G}_i := \sigma(X_t, t \leq \max_{j<i} \tau_{\sigma_j})$. Also, since σ is random, $P(\mathcal{B}_i) = 1/i$.

For a subset $Q \subset S$, we set $\tau_Q = \max_{i\in Q} \tau_i$. Now,

$$\tau_{\{\sigma(1),\ldots,\sigma(i)\}} = \tau_{\{\sigma(1),\ldots,\sigma(i-1)\}} + \mathbf{1}_{\mathcal{B}_i} R_i \tag{2}$$

where $R_i = \tau_{\{\sigma(1),\ldots,\sigma(i)\}} - \tau_{\{\sigma(1),\ldots,\sigma(i-1)\}}$. Note that $E[R_i|\mathcal{G}_{i-1}] \leq h_{\max}$, and therefore, since $\mathcal{B}_i$ is measurable on $\mathcal{G}_{i-1}$,

$$E[R_i \mathbf{1}_{\mathcal{B}_i}] = E[E[R_i \mathbf{1}_{\mathcal{B}_i} |\mathcal{G}_{i-1}]] \leq h_{\max} P(\mathcal{B}_i) = \frac{h_{\max}}{i}.$$

Combining the last display with (2) and summing, one obtains the claim. $\square$

Remark 1.2. The bound is particularly effective if $h_{\max}$ is not too far from $H(i, j)$ for "most" $j \neq i$. This is e.g. the case in the classical coupon collector problem (i.e., where G is the complete graph and all weights equal 1).

Exercise 1. *Derive a version of Matthew's bound for continuous time chains with unit rate of jump at each state (i.e., $q_{xx} = 1$ for all $x \in S$).*

Exercise 2. *Using Matthew's bound, show that for SRW on the two dimensional torus of side N, $ET_{\rm COV} \leq n^2(\log n)^2$.*

2. An upper bound on cover times via RK2

We return to the setting of continuous time chains on a finite graphs with a distinguish vertex x_0. We introduce a slightly different notion of cover time, namely

$$\tau_{\rm COV} = \min\{u > 0 : \ell_x(\theta_u) > 0, \forall x \in V\}.$$

In words, every vertex of V has been visited before a local time of u has been accumulated at the origin. While this definition seems unrelated to $t_{\rm COV}$, we will later see that one can transfer estimates between the two.

Fix now t and suppose that at time θ_t, there exists a vertex x so that $\ell_x(\theta_t) = 0$. For that x, the left side of the GRK2 theorem is

$$\ell_x(\theta_t) + \frac{1}{2}\phi_x^2 = \frac{1}{2}\phi_x^2. \tag{3}$$

Note that x depends only on the Markov chain, not on the GFF.

Fix $\epsilon > 0$. Assume that t is such that

$$P(\exists x : \ell_x(\theta_t) = 0) = P(\tau_{\rm COV} > t) > \epsilon. \tag{4}$$

Let $\bar{G} = \max_x G(x,x)$. Fix $K = K(\epsilon)$ so that

$$\max_{x \in V} P(\frac{1}{2}\phi_x^2 > K(\epsilon)\bar{G}) \leq \epsilon/2. \tag{5}$$

(By Gaussian scalling, $K(\epsilon)$ does not depend on $|V|$ or on the graph structure.)

Combining the last display with (3) and (4), we get that for this t,

$$P(\exists x \in V : \ell_x(\theta_t) + \frac{1}{2}\phi_x^2 < K(\epsilon)\bar{G}) > \frac{\epsilon}{2}. \tag{6}$$

Now we use the GRK2 theorem: the left side of (6) equals

$$P(\exists x \in V : \frac{1}{2}(\phi_x + \sqrt{2t})^2 < K(\epsilon)\bar{G}) \tag{7}$$

and therefore, the expression in (7) is larger than $\epsilon/2$. On the other hand, if $t > K(\epsilon)\bar{G}$ then

$$\begin{aligned} P(\exists x \in V : (\phi_x + \sqrt{2t})^2 < 2K(\epsilon)\bar{G}) &\leq P(\min_{x \in V} \phi_x < -\sqrt{2t} + \sqrt{2K(\epsilon)\bar{G}}) \\ &= P(\max_{x \in V} \phi_x > \sqrt{2t} - \sqrt{2K(\epsilon)\bar{G}}) =: Q(t,\epsilon). \end{aligned}$$

In particular, if t is chosen such that $Q(t,\epsilon) < \epsilon/2$ one obtains a contradiction. We have just proved:

Proposition 2.1. *Fix $\epsilon > 0$ and let $K(\epsilon)$ be as in (5). If t is chosen so that*

$$P(\max_{x \in V} \phi_x > \sqrt{2t} - \sqrt{2K(\epsilon)\bar{G}}) < \epsilon/2$$

then

$$P(\tau_{\rm COV} > t) < \epsilon.$$

Before moving on, we are going to use a general Gaussian tool to make the estimate of Proposition 2.1 more explicit. Let $M := E\max_{v\in V}\phi_x$. Recall that by the Borell–Tsirelson-Ibragimov-Sudakov inequality,

$$P(\max_{v\in V}\phi_x > M + y) \le 2e^{-y^2/2\bar G}.$$

Substituting this in Proposition 2.1 gives the following.

Theorem 2.2. *There exists a universal function $C(\epsilon)$ so that*

$$P(\sqrt{2\tau_{\rm cov}} > E\max_{v\in V}\phi_x + C(\epsilon)\sqrt{\bar G}) < \epsilon.$$

Exercise 3. *Instead of using Borell's inequality, use the inequality*

$$P((\phi_x+\sqrt{2t})^2 < 2K(\epsilon)\bar G) \le P(\phi_x < -\sqrt{2t} + \sqrt{2K(\epsilon)\bar G}) \le Ce^{-(\sqrt{2t}-\sqrt{K(\epsilon)\bar G})^2/2\bar G}$$

and a union bound to deduce a bound of the form

$$P(\tau_{\rm cov} > \bar G\log|V| + C(\epsilon)\bar G\sqrt{\log|V|}) < \epsilon.$$

One may wonder what is the relation between $\tau_{\rm cov}$ and $t_{\rm cov}$. Luckily, this is not hard to evaluate. Indeed, let T_0 denote the time it takes the Markov chain to leave x_0 and then return time to x_0. By a variant of Kac's lemma, $ET_0 = 1/(\lambda_0 p_{x_0})$ where p_{x_0} is the stationary distribution of the chain, and (as can be checked from detailed balance), $p_{x_0} = 1/|V|$. On the other hand, the time spent during one excursion at x_0 has mean $1/\lambda_{x_0}$. To accumulate local time t at the origin thus is expected to require $t\lambda_0$ such excursions, which are expected to take $t\lambda_0\cdot|V|/\lambda_0 = t|V|$. Thus, one *expects* that $t_{\rm cov}\sim\tau_{\rm cov}\cdot|V|$.

To make the above precise requires some concentration. The main tool is the following lemma. Let $\tilde G_0$ denote the Green function of the walk killed when hitting x_0

Lemma 2.3 (Kac moment formula)**.** *For $x\ne x_0$,*

$$E^xT_0^2 = 2\sum_{z\ne x_0}(\tilde G\cdot\tilde G)(x,z).$$

Proof. We have that

$$\begin{aligned} E^xT_0^2 &= E^x(\int_0^\infty 1_{s<T_0}ds)^2 = E^x\int_0^\infty\int_0^\infty 1_{s<T_0}1_{t<T_0}dsdt \\ &= 2E^x\int_0^\infty\int_t^\infty 1_{s<T_0}1_{t<T_0}dsdt \\ &= 2\int_0^\infty\int_0^\infty\sum_{w\in U}\tilde P^x(X_t=w)\tilde P^w(s<T_0)dsdt \\ &= 2\sum_{z\ne x_0}(\tilde G\cdot\tilde G)(x,z), \end{aligned}$$

where $\tilde P$ is the law of the process killed upon hitting x_0. ☐

In particular, if $t^0_{\rm hit} := \max_{x\in U}E^x(T_0)$ then

$$E^x(T_0^2) \le 2E^x(T_0)\cdot t^0_{\rm hit}. \tag{8}$$

Let T_0^i denote the successtive return times to x_0, which form an iid sequence. Write $T_0^i = \tau_0^i + S_0^i$ where τ_0^i are the escape times from x_0; τ_0^i are iid exponential with parameter λ_{x_0}. Fix N. Then,

$$E((\sum_{i=1}^N T_0^i)^2) = N^2 E(T_0)^2 + N\left(E((T_0)^2) - (ET_0)^2\right).$$

However, using (8),

$$ES_0^2 = \sum_{z \in V} \frac{\lambda_{x_0,z}}{\lambda_{x_0}} E^z(S_0^2) \leq 2E(S_0) \cdot t^0_{\text{hit}}.$$

It follows that

$$ET_0^2 - (ET_0)^2 = E(S_0^2) - (ES_0)^2 + E\tau_0^2 - (E\tau_0)^2 \leq \frac{1}{\lambda_0^2} + ES_0(t^0_{\text{hit}} - ES_0).$$

It follows that

$$P\left(\left|\frac{\sum_{i=1}^N T_0^i}{NET_0} - 1\right| > \delta\right) \leq \frac{1}{\delta^2 N}\left(\frac{1}{\lambda_0^2} + \frac{2t^0_{\text{hit}}}{ES_0}\right).$$

In particular, if $t^0_{\text{hit}} \ll \lambda_0 \tau_{\text{cov}} ET_0 = \tau_{\text{cov}}|V|$ then, taking $N = \tau_{\text{cov}}\lambda_0$, one concludes (using the steps described above) that $t_{\text{cov}} \sim \tau_{\text{cov}}|V|$.

Exercise 4. *Fill in the details to show that $t^0_{\text{hit}} \ll t_{\text{cov}}$ then $t_{\text{cov}} \sim \tau_{\text{cov}}|V|$.*

3. A remark on positivity

Consider Brownian motion B_t and let $T_1 = \min\{t : B_t = 1\}$. We have the following.

Lemma 3.1. *Almost surely, $L^x(T_1) > 0$ for any $x \in [0,1)$.*

That the claim is true for fixed x is obvious from the downcrossing representation. The point is that we claim this to be true for all x at once (and not merely for almost every x).

The claim follows at once from the first Ray-Knight theorem:

Theorem 3.2 (First Ray-Knight theorem)**.** *Let $L^{x,a}(t)$ be the local time of Brownian motion started at $a > 0$ and let $W^{(2)}$ denote a two dimensional Brownian motion started at 0. Let $T_0 = \min\{t : B_t = 0\}$. Then*

$$\{L^{x,a}(T_0)\}_{x\in[0,a]} \stackrel{d}{=} \{|W_x^{(2)}|^2\}_{x\in[0,a]}.$$

Indeed, Lemma 3.1 follows by noting that 0 is polar for the two dimensional Brownian motion.

The proof of RK1 is not hard, given the downcrossing representation, see the Mörters-Peres book. We provide instead a sketch of an alternative proof of Lemma 3.1 that avoids the use of RK1. Recall that by the second Ray-Knight theorem gave that $L^x(\theta_u)$ is a $BESQ^0(\sqrt{u})$ process, denoted Q_x. Since $Q_x > 0$ for $x < X_0$ and $Q_x = 0$ for $x \geq X_0$, it follows that $L^x(\theta_u)$ is positive on $[0, X_0)$. We claim that $X_0 = \max_{t \leq \theta_u} B_t =: X_1$. Indeed, clearly $X_1 \geq X_0$. If $X_1 > X_0$ then $L^x(\theta_u) = 0$ on (X_0, X_1), which implies by the integral representation that the occupation measure of (X_0, X_1) vanishes. This is impossible for Brownian motion since it is continuous almost surely and hits X_1. From here, the proof of Lemma 3.1 is not far.

4. Generalized second Ray-Knight theorem on metric graphs

We return to the discrete setup of Lecture 1. We now replace each edge (x, y) by an interval of length $W_{x,y}$, and thus obtain a metric space $\hat{V}$. Note that V is naturally embedded in $\hat{V}$. The construction follows T. Lupu.

We now define a Brownian motion on $\hat{V}$. Some care is needed in the construction; one way is to use Ito's excursion construction of Brownian motion. Another is to divide each edge to intervals of length $1/N$, attach to each a weight $W_{x,y} \cdot N$, and consider the random walk with these weights on $\hat{V}$, and finally take weak limits.

In what follows, we use B_t to denote the Brownian motion on $\hat{V}$. Let T_i be the successive hitting times of V by $\{B_t\}$: $T_0 = 0$ and $T_{i+1} = \inf\{t > T_i : B_t \in V, B_t \neq X_{T_i}\}$. For $t \geq 0$, let $A_t = \max\{T_i : T_i \leq t\}$ and set $X_t = B_{A_t}$.

Lemma 4.1. *Conditioned on $X_{T_{i-1}} = x$, the local time $\hat{L}_x(T_i) - \hat{L}_x(T_{i-1})$ is exponentially distributed with parameter W_x. Further,*

$$P(X_{T_i} = y | X_{T_{i-1}} = x) = \frac{W_{xy}}{W_x}.$$

Proof. We use the representation as a weak limit of random walks: conditioned on exiting x in the direction of y, the probability of completing the excursion before returning to x is $1/N$. The number of excursions on the (x, y) edge needed before completion of a full excursion is Geometric with parameter N. Since by construction one choses the (x, y) edge with probability $W_{x,y}/W_x$, one immediately obtains the second claim. To see the claim concerning $\hat{L}_x(T_i) - \hat{L}_x(T_{i-1})$, let N_y denote the number of excursions on the (x, y) edge needed before y is hit. Then the total time accumulated at x before hitting y is $\sum_{i=1}^{N_y} E_i(x, y)$ where $E_i(x, y)$ are iid exponentials of parameter NW_{xy}, which can be written as $(NW_{x,y})^{-1}\hat{E}_i(x, y)$ where $\hat{E}_i(x, y)$ are iid exponential of parameter 1. Thus, the running time before a succesful excursion occurs is

$$\frac{1}{NW_{xy}} \sum_{i=1}^{N_y} \hat{E}_i(x, y) \xrightarrow{d} \mathcal{E}_{x,y} \sim Exp(W_{xy}).$$

Finally, $\hat{L}_x(T_i) - \hat{L}_x(T_{i-1})$, conditioned on $X_{T_{i-1}} = x$, is distributed as $\min_{y \neq x} \mathcal{E}_{x,y}$, which is exponential of parameter W_x. $\square$

Thus, the process $\{X_t\}_{t\geq 0}$ is a continuous time Markov chain on V with rate matrix W.

The Brownian motion on $\hat{V}$ has local times, which we denote $\{\hat{L}^x(t)\}_{x\in\hat{V}}$. The construction is essentially the same as for Brownian motion.

Recall that the downcrossing representation of local time of Brownian motion implied that $\hat{L}^0(T_1)$ is exponentially distributed. Using a similar argument, one shows the following.

Exercise 5. *The process $\{\hat{L}_x(T_i)\}_{x\in V, i\in\mathbb{Z}_+}$ has the same law as the process $\{\ell^x(T_i)\}_{x\in V, i\in\mathbb{Z}_+}$. Further, defining $\theta_u(B)$ and $\theta_u(X)$ in the natural way, we have that*

$$\{\hat{L}_x(\theta_u(B))\}_{x\in V} \stackrel{d}{=} \{\ell^x(\theta_u(X))\}_{x\in V}.$$

An important observation, consequence of Lemma 3.1, is that if $\hat{L}_z(\theta_u) > 0$ for $z = x$ and $z = y$ and the edge (x, y) has been traversed by the random walk then $\hat{L}_z(\theta_u) > 0$ for $z \in (x, y)$.

To complete our discussion of the Brownian motion on the metric graph, we should discuss the GFF $\hat{\phi}_x$ there. It is not hard to see that conditioned on $\{\hat{\phi}_x\}_{x\in V}$, we have that $\{\hat{\phi}_z\}_{z\in(x,y)}$ has the law of a Brownian bridge of length $W_{x,y}$, with endpoints $(\hat{\phi}(z), \hat{\phi}(y))$. The GRK2 theorem now reads

$$\{\hat{L}_z(\theta_u) + \frac{1}{2}(\hat{\phi}'_z)^2\}_{z\in\hat{V}} \stackrel{d}{=} \{\frac{1}{2}(\hat{\phi}_z + \sqrt{2u})^2\}_{z\in\hat{V}}. \tag{9}$$

Here $\hat{\phi}'$ is a copy of the GFF independent of the Brownian motion B.

5. Consequences for cover times

We follow the ideas of A. Zhai. Use the left side of (9) to define a GFF ϕ in two steps: first, define $M_u = |\hat{\phi} - \sqrt{2u}|$ using the formula, then sample the sign of M_u according to the conditional law.

By construction, if M_u vanishes anywhere along a edge (x, y) then necessary $\hat{L}(\theta_u)$ vanishes there. By the positivity claim, this means that the edge (x, y) was not traversed. We thus conclude that necessarily, all vertices where $\hat{L}_z(\theta_u) > 0$ (which are a connected set!) satisfy that the sign of $\hat{\phi}_z + \sqrt{2t}$ is the same; in particular, since the vertices with $\hat{L}_z(\theta_u) > 0$ belong to a connected cluster of x_0, necessarily $\hat{\phi}_z + \sqrt{2t} > 0$ there. Thus we have constructed a coupling between $\hat{L}_z(\theta_u)$ and the GFF ϕ which has the following property: if $L_z(\theta_u) > 0$ for all $z \in V$, then $\hat{\phi}_z + \sqrt{2t} > 0$, i.e. $\min_{z\in V} \phi_z > -\sqrt{2t}$. In particular, we have shown that

$$P(\tau_{\text{cov}} < t) \leq P(\max_{z\in V} \phi_z < \sqrt{2t}).$$

Exercise 6. *(*) Derive the sharpest upper and lower probability bound (e.g. exponential tails) you can on cover times. You can express bounds in terms of relevant bounds on the GFF, which you do not need to prove.*