LECTURE 1: GENERALIZED RAY KNIGHT THEOREM FOR FINITE MARKOV CHAINS

We will work with a continuous time reversible Markov chain X_t on a finite connected state space, with generator $Lf(x) = \sum_y q_{x,y} f(y)$. (Recall that $q_{x,x} = -\sum_{y \neq x} q_{x,y}$.) We will be interested in the local time process of X_t (defined below) and its relation with certain Gaussian fields.

1. Preliminaries - Ito's formula

We begin with a couple of preparatory lemmas.

Lemma 1.1 (Ito's formula)**.** *Notation as above. For any function f,*

$$f(X_t) = f(X_0) + \int_0^t Lf(X_s)ds + M_t$$

where M_t is a martingale.

Proof. Fix $u > t$. Let $\mathcal{F}_t$ be the cannonical filtration. Note that for δ small,

$$E[f(X_{u+\delta})|\mathcal{F}_u] = f(X_u) + \delta \sum_y q_{X_u,y} f(y) + O(\delta^2)$$

where the $O(\delta^2)$ comes from the possibility of having two or more jumps and is uniform. Therefore, we have

$$\begin{aligned} E[f(X_{u+\delta})|\mathcal{F}_t] &= E[E[f(X_{u+\delta})|\mathcal{F}_u]|\mathcal{F}_t] \\ &= E[f(X_u)|\mathcal{F}_t] + \delta E[Lf(X_u)|\mathcal{F}_t] + O(\delta^2). \end{aligned}$$

(The term $O(\delta^2)$ is uniform and deterministic.)

On the other hand,

$$E[\int_0^{u+\delta} Lf(X_s)ds|\mathcal{F}_t] = E[\int_0^u Lf(X_s)ds|\mathcal{F}_t] + \delta E[Lf(X_u)|\mathcal{F}_t] + O(\delta^2).$$

Define $Z_u = E[f(X_u) - \int_0^u Lf(X_s)ds|\mathcal{F}_t]$. Then, from the above, $Z_{u+\delta} = Z_u + O(\delta^2)$, i.e. $dZ_u/du = 0$ for all $u \geq t$. Hence, $Z_u = Z_t$, for $u \geq t$, which proves the claim. $\square$

Lemma 1.2 (Exponential Ito's formula)**.** *Notation as above. For any function f with $Lf = g$ and $\min |f| > 0$,*

$$f(X_t) \exp\left(-\int_0^t \frac{g(X_s)}{f(X_s)} ds\right)$$

is a martingale.

Proof. Write

$$\begin{aligned}
Z_{t+\delta} &= f(X_{t+\delta})\exp\left(-\int_0^t \frac{g(X_s)}{f(X_s)}ds\right)\exp\left(-\int_t^{t+\delta}\frac{g(X_s)}{f(X_s)}ds\right) \\
&= Z_t \exp\left(-\int_t^{t+\delta}\frac{g(X_s)}{f(X_s)}ds\right) \\
&\quad +[f(X_{t+\delta})-f(X_t)]\exp\left(-\int_0^t \frac{g(X_s)}{f(X_s)}ds\right)+O(\delta^2) \\
&= Z_t - Z_t\int_t^{t+\delta}\frac{Lf(X_s)}{f(X_s)}ds \\
&\quad +[f(X_{t+\delta})-f(X_t)]\exp\left(-\int_0^t \frac{Lf(X_s)}{f(X_s)}ds\right)+O(\delta^2).
\end{aligned}$$

Taking conditional (on $\mathcal{F}_t$) expectation, we get

$$\begin{aligned}
E[Z_{t+\delta}|\mathcal{F}_t] &= Z_t - \delta Z_t\frac{Lf(X_t)}{f(X_t)} \\
&\quad +\delta Lf(X_t)\exp\left(-\int_0^t \frac{g(X_s)}{f(X_s)}ds\right)+O(\delta^2) \\
&= Z_t - \delta Z_t\frac{Lf(X_t)}{f(X_t)}+\delta Z_t\frac{Lf(X_t)}{f(X_t)}+O(\delta^2) \\
&= Z_t + O(\delta^2).
\end{aligned}$$

From here, proceed as in the proof of Ito's lemma. □

Exercise 1. *Show that the same proof works if instead of X_t we consider the Markov process (X_t, Y_t) where $Y_t = Y_0 + \int_0^t h(X_s)ds$, replacing the generator L by $\tilde{L}f(x,z) = Lf(x,z) + h(x)\frac{\partial f}{\partial z}(x,z)$. This remark will be important in what follows.*

2. Green functions

We now consider our Markov chains on graphs $G = (V, E)$ with no self loops. We interpret the q_{xy} with $x \neq y$ as conductances $W_{x,y} > 0$ on the edges, and set $\lambda_x = \sum_{y\neq x} W_{x,y}$. Thus $q_{x,x} = -\lambda_x$. Introduce the Dirichlet form

$$\mathcal{E}(f,f) = \frac{1}{2}\sum_{x,y\in V} W_{x,y}(f(y)-f(x))^2.$$

Fix $x_0 \in V$ and set $U = V \setminus x_0$. Let $\tau_0 = \inf\{t : X_t = x_0\}$. We consider the chain killed at time τ_0, and introduce the *Green function*

$$g(x,y) = E^x\int_0^{\tau_0}\mathbf{1}_{X_t=y}dt =: E^x\ell_y(\tau_0)$$

where $\ell_y(t) = \int_0^t \mathbf{1}_{X_s=y}ds$ is the *local time* of the Markov chain at y.

Lemma 2.1. *g is a symmetric, positive definite matrix.*

Proof. Consider the discrete time process $Y_n = X_{\tau_n}$ where τ_n are the jump times. This is of course a Markov chain with jump probabilities $P_{x,y} = W_{x,y}/\lambda_x$. Let $N_0 = \min\{n : Y_n = x_0\}$. Let τ_y^+ denote a jump time from y (i.e., exponential with parameter λ_y). Let $U(n,x,y)$ denote the collection of paths $(z_0 = x, \ldots, z_n = y)$ entirely inside U. Then, by the strong Markov property,

$$\begin{aligned} E^x \int_0^{\tau_0} \mathbf{1}_{X_t=y} &= \sum_{n=0}^{\infty} P^x(Y_n = y, n < N_0) E^y(\tau_y^+) = \frac{1}{\lambda_y} \sum_{n=0}^{\infty} P^x(Y_n = y, n < N_0) \\ &= \frac{1}{\lambda_y} \sum_{n=0}^{\infty} \sum_{\bar{z} \in U(n,x,y)} \frac{W(x,z_1)}{\lambda_x} \frac{W(z_1,z_2)}{\lambda_{z_1}} \cdot \frac{W(z_{n-1},y)}{\lambda_{z_{n-1}}} \end{aligned}$$

Reversing the steps (using that $W_{x,y}$ is symmetric), we obtain that this equals $E^y \int_0^{\tau_0} \mathbf{1}_{X_t=x}$, proving that $g(x,y) = g(y,x)$.

To see that g is positive definite, note that the computation above shows that the Green function is related to the discrete Green function

$$G_D(x,y) = \frac{1}{\lambda_y} \sum_{n=0}^{\infty} P(Y_n = y, n < N_0) = \frac{1}{\lambda_y} \sum_{n=0}^{\infty} \hat{P}^n(x,y),$$

where $\hat{P}$ is the transition matrix of the discrete chain restricted to U. Since G is connected, $\hat{P}$ is sub-stochastic and its top eigenvalue is strictly less than 1. Therefore, $G_D(x,y) = \frac{1}{\lambda_y}(I - \hat{P})^{-1}$. It follows that all eigenvalues are strictly positive. $\square$

We note that if $\hat{L}$ is the generator restricted to U then $G_D = (-L)^{-1}$. Indeed, let Λ be the diagonal matrix with $\Lambda_{xx} = \lambda_x$, note that $-L = \Lambda(I - \hat{P})$ and therefore $(-L)^{-1} = (I - \hat{P})^{-1}\Lambda^{-1}$ and therefore

$$(-L)^{-1}(x,y) = (I + \hat{P} + \hat{P}^2 + \cdots)(x,y) \cdot \frac{1}{\lambda_y} = G_D(x,y).$$

Exercise 2. *Show that $\mathcal{E}(f,f) = \langle f, Lf \rangle$ where the inner product is in $\ell^2(V)$.*

3. The Gaussian free field associated on G

Since g is symmetric and positive definite, we can associate to it a centered Gaussian field $\{\phi_x\}_{x \in V}$ by setting $\phi_{x_0} = 0$ and $E\phi_x\phi_y = g(x,y)$ for $x, y \in U$. Since $g = (-L)^{-1}$, we see from Exercise 2 that the Gaussian density is proportional to $\exp(-\frac{1}{2}\mathcal{E}(\phi,\phi))$ (with $\phi_{x_0} = 0$).

4. The generalized second Ray-Knight theorem

Set $\theta_u = \inf\{t \geq 0 : \ell_{x_0}(t) > u\}$, the inverse local time at x_0.

Theorem 4.1. *Let $\{X_t\}$ and $\{\phi_x\}$ be independent, as above.*

$$\left(\ell_x(\theta_u) + \frac{1}{2}\phi_x^2\right)_{x \in V} \stackrel{d}{=} \left(\frac{1}{2}(\phi_x + \sqrt{2u})^2\right)_{x \in V}.$$

The theorem has a long history, going back to Ray and to Knight (in the Brownian motion case), passing through some results of Dynkin that we will describe later, hopefully, and of Eisenbaum. This version is due to Eisenbaum, Kaspi, Marcus, Rosen and Shi.

Proof. The proof we present is due to Sabot and Tarres. Throughout, we write E^x for expectation with respect to the Markov chain started at x and $\mathbb{E}^x$ for expectation with respect to both the Markov chain and the Gaussian field.

Fix $h > 0$ a positive bounded measurable function (on $\mathbb{R}^V$). Introduce the notation

$$d\varphi = \delta_{\varphi_{x_0}} \prod_{x\in U} d\varphi_x, \quad C = (2\pi)^{-|U|/2} (\det g)^{-1/2}.$$

We write

$$\begin{aligned} \mathbb{E}^{x_0}\left(h((\ell_x(\theta_u) + \frac{1}{2}\phi_x^2)_{x\in V}\right) &= CE^{x_0} \int_{\mathbb{R}^V} h(\ell_x(\theta_u)) + \frac{1}{2}\varphi_x^2)_{x\in V}) e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} d\varphi \\ (1) \qquad &= CE^{x_0} \sum_{\sigma\in\Sigma_V} \int_{\mathbb{R}_+^U} h(\ell_x(\theta_u)) + \frac{1}{2}\varphi_x^2)_{x\in V}) e^{-\frac{1}{2}\mathcal{E}(\sigma\varphi,\sigma\varphi)} d\varphi \end{aligned}$$

where $\Sigma_V = \{\sigma \in \{-1,1\}^V : \sigma_{x_0} = 1\}$.

Define (with obvious vector notation) $\Phi = \sqrt{2\ell(\theta_u) + \varphi^2}$ and set

$$D_u = \{\Phi \in \mathbb{R}_+^V : \Phi_{x_0} = \sqrt{2u}, \Phi_x^2/2 \geq \ell_x(\theta_u), \forall x \in U.\}$$

Then $\Phi : \mathbb{R}_+^V \cap \{\varphi_{x_0} = 0\} \to D_u$ has inverse given by $\varphi_x = \sqrt{\Phi_x^2 - 2\ell_x(\theta_u)}$. This is a diagonal transformation with Jacobian of the inverse given by

$$J = \prod_{x\in U} \frac{\Phi_x}{\varphi_x}.$$

Making now the change of variables, we get that the right side of (1) equals

$$(2) \qquad CE^{x_0} \sum_{\sigma\in\Sigma_V} \int_{\mathbb{R}_+^U} h((\frac{\Phi_x^2}{2})_{x\in V}) e^{-\frac{1}{2}\mathcal{E}(\sigma\varphi,\sigma\varphi)} \prod_{x\in U} \frac{\Phi_x}{\varphi_x} \mathbf{1}_{\Phi\in D_u} d\Phi.$$

(In the last display, we abused notation by disregarding the value of Φ_{x_0}, since it equals $\sqrt{2u}$ under the integration.)

Fix now $\Phi \in \mathbb{R}_+^V \cap \{\Phi_{x_0} = \sqrt{2u}\}$, and define the stopping time

$$T = \inf\{t \geq 0 : \ell_x(t) = \frac{1}{2}\Phi_x^2, \text{ some } x \in V\}.$$

Set, for $t \leq T$, $\Phi_x(t) = \sqrt{\Phi_x^2 - 2\ell_x(t)}$. This is a means to introduce dynamics (in reverse time!). Note that

$$\Phi \in D_u \Leftrightarrow X_T = x_0 \Leftrightarrow T = \theta_u,$$

and that in that case, $\varphi = \Phi(\theta_u)$.

Now we exploit the dynamics: set

$$M_t^{\sigma\Phi} = e^{-\frac{1}{2}\mathcal{E}(\sigma\Phi(t),\sigma\Phi(t))} \frac{\prod_{x\neq x_0} \sigma_x \Phi_x(0)}{\prod_{x\neq X_t} \sigma_x \Phi_x(t)}.$$

Note that the product in the right side, at time T, could be written as

$$\prod_{x\neq x_0} \sigma_x \Phi_x / \prod_{y\neq X_t} \sigma_y \varphi_y,$$

which resembles the Jacobian of the change of variables.

The heart of the proof is the following lemma, whose proof we postpone.

Lemma 4.2. *The process $(M_{t\wedge T}^{\sigma\Phi})_{t\geq 0}$ is a uniformly integrable martingale.*

We will also need the following easy lemma. Set $N_t^\Phi = \sum_{\sigma\in\Sigma_V} M_t^{\sigma\Phi}$.

Lemma 4.3. *For all $x \neq x_0$, $N_T^\Phi \mathbf{1}_{\{X_T = x\}} = 0$.*

Proof. Let σ^x equal σ except that the spin in the x location is flipped. Note that $\Phi_x(T) = 0$ if $X_T = x$, by definition. On the latter event we therefore have that $\sigma^x\Phi(T) = \sigma\Phi(T)$. On the other hand, $(\sigma^x\varphi)_x = \sigma^x\Phi_x(0) = -\sigma\Phi_x(0)$. Hence, $M_T^{\sigma^x\Phi} = -M_T^{\sigma\Phi}$ on the event $X_T = x$. Since

$$N_T^\Phi = \sum_{\sigma\in\Sigma_V} M_T^{\sigma\Phi} = \sum_{\sigma\in\Sigma_V} M_T^{\sigma^x\Phi},$$

we obtain that

$$N_T^\Phi \mathbf{1}_{\{X_T=x\}} = \frac{1}{2}\sum_{\sigma\in\Sigma_V}(M_T^{\sigma\Phi} + M_T^{\sigma^x\Phi})\mathbf{1}_{\{X_T=x\}} = 0.$$

□

We can now complete the proof. Note that the expression in (2) equals

$$\begin{aligned} C\int_{\mathbb{R}_+^U} h((\frac{\Phi_x^2}{2})_{x\in V})E^{x_0}[N_T^\Phi \mathbf{1}_{X_T=x_0}]d\Phi &= C\int_{\mathbb{R}_+^U} h((\frac{\Phi_x^2}{2})_{x\in V})E^{x_0}[N_T^\Phi]d\Phi \\ &= C\int_{\mathbb{R}_+^U} h((\frac{\Phi_x^2}{2})_{x\in V})N_0^\Phi d\Phi, \end{aligned}$$

where the first equality is due to Lemma 4.3 and the second to Lemma 4.2. The last expression then equals

$$C\int_{\mathbb{R}_+^U} h((\frac{\Phi_x^2}{2})_{x\in V})\sum_{\sigma\in\Sigma_V} e^{-\frac{1}{2}\mathcal{E}(\sigma\Phi,\sigma\Phi)}d\Phi = C\int_{\mathbb{R}^U} h((\frac{\Phi_x^2}{2})_{x\in V})e^{-\frac{1}{2}\mathcal{E}(\Phi,\Phi)}d\Phi.$$

Since $\Phi_{x_0} = \sqrt{2u}$, we see that Φ under the last measure is distributed like $\phi + \sqrt{2u}$. This completes the proof. □

It remains to prove Lemma 4.2.

Proof of Lemma 4.2. We introduce the process $\tilde{X}_t = (X_t, \ell_t)$ with generator $\tilde{L} = (L + \frac{\partial}{\partial \ell_x})$. Set

$$f(x,\ell) = \prod_{y\neq x} \frac{1}{\sigma_y\sqrt{\Phi_y^2 - 2\ell_y}}.$$

Recall that

$$M_t^{\sigma\Phi} = e^{-\frac{1}{2}\mathcal{E}(\sigma\Phi(t),\sigma\Phi(t))}\frac{\prod_{x\neq x_0}\sigma_x\Phi_x(0)}{\prod_{x\neq X_t}\sigma_x\Phi_x(t)}. \tag{3}$$

Note that, since $\Phi_x(t) = \sqrt{\Phi_x^2 - 2\ell_x(t)}$, we have $d\Phi_x(t)/dt = -\mathbf{1}_{\{X_t=x\}}/\Phi_x(t)$ and therefore

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\sigma\Phi(t),\sigma\Phi(t)) &= \frac{1}{2}\frac{d}{dt}\sum_{x,y} W_{x,y}(\sigma_x\Phi_x(t) - \sigma_y\Phi_y(t))^2 \\ &= \frac{2}{(\sigma\Phi(t))_{X_t}} L(\sigma\Phi(t))(X_t). \end{aligned}$$

On the other hand,

$$\frac{Lf(X_t,\ell_t)}{f(X_t,\ell_t)} = \frac{\sum_{y\neq X_t} W_{X_t,y}(f(y,\ell_t)-f(X_t,\ell_t))}{f(X_t,\ell_t)}$$
$$= \sum_{y\neq X_t} W_{X_t,y}\left(\frac{f(y,\ell_t)}{f(X_t,\ell_t)}-1\right) = \sum_{y\neq X_t} W_{X_t,y}\left(\frac{(\sigma\Phi(t))_y}{(\sigma\Phi(t))_{X_t}}-1\right) = \frac{L(\sigma\Phi(t))(X_t)}{(\sigma\Phi(t))(X_t)}.$$

Combining the last two displays we get that

$$\frac{d}{dt}\mathcal{E}(\sigma\Phi(t),\sigma\Phi(t)) = 2\frac{Lf(X_t,\ell_t)}{f(X_t,\ell_t)} = 2\frac{\tilde{L}f(X_t,\ell_t)}{f(X_t,\ell_t)},$$

where the second equality is due to the fact that $f(x,\ell)$ does not depend on ℓ_x. Substituting in (3), we conclude that

$$\frac{M_t^{\sigma\Phi}}{M_0^{\sigma\Phi}} = \frac{f(X_t,\ell_t)}{f(x_0,0)} e^{-\int_0^t \frac{\tilde{L}f}{f}(X_s,\ell_s)ds}.$$

Applying Lemma 1.2, we conclude that $M^{\sigma\Phi}_{t\wedge T_n}$ is a martingale, where $T_n = \inf\{t : \ell_x(t) \geq \frac{1}{2}\Phi_x^2 - \frac{1}{n}$ for some $x \in V\}$. Clearly, $T_n \to T$. We will prove that $M_{t\wedge T_n}$ is uniformly (in n and t) integrable, which then implies that $M_{t\wedge T}$ is a uniformly integrable martingale.

Toward the latter, a direct computation similar to the above gives that

$$\left|\frac{\tilde{L}f}{f}(x,\ell)\right| = \left|\sum_{y\neq x} W_{x,y}\frac{\sigma_y\sqrt{\Phi_y^2-2\ell_y}-\sigma_x\sqrt{\Phi_x^2-2\ell_x}}{\sigma_x\sqrt{\Phi_x^2-2\ell_x}}\right|.$$

Therefore,

$$\left|\frac{\tilde{L}f}{f}\right|(X_t,\ell_t) \leq \frac{C(W,\Phi)}{\Phi_{X_t}(t)}.$$

Thus,

$$\begin{aligned}
\left|\int_0^t \frac{\tilde{L}f}{f}(X_s,\ell_s)ds\right| &\leq C(W,\Phi)\int_0^t \frac{1}{\Phi_{X_s}(s)}ds \\
&= \sum_{x\in V} C(W,\Phi)\int_0^{\ell_x(t)} \frac{1}{\sqrt{\Phi_x^2-2\ell}}d\ell \\
&= \sum_{x\in V} C(W,\Phi)\int_{\Phi_x^2}^{\Phi_x^2(t)} \frac{1}{\sqrt{\ell}}d\ell \\
&= 2\sum_{x\in V} C(W,\Phi)(\sqrt{\Phi_x^2}-\sqrt{\Phi_x^2(t)}) \leq C(W,\Phi,|V|).
\end{aligned}$$

Another observation is that since $|V| < \infty$ and $\sum_{x\in V}\ell_x(t) = t$, we have that $T \leq C(\Phi,|V|)$. In particular, combined with the last display we conclude that, for $t \leq T$,

$$M_t^{\sigma\Phi} \leq C(\Phi,|V|,W)|f(X_t,\ell_t)|. \tag{4}$$

So it only remains to show that $f(X_{t\wedge T},\ell_{t\wedge T})$ is uniformly integrable. Toward this end, run the chain X_t until time S at which $\ell_x(S) \geq \Phi_x^2/2$ for all x. Clearly, $S < \infty$ since V is connected. Let $T_x = \inf\{t : \ell_x(t) = \Phi_x^2 2/2\}$ and let s_x denote the time accumulated in the last visit to x before t. Note that if $x \neq X_t$ then

$\Phi_x^2 - 2\ell_x(t \wedge T) \geq 2s_x$. On the other hand, the variables s_x are independent, exponentially distributed with parameter λ_x. We then get that

$$|f(X_t, \ell_t)| \leq \prod_{x \neq X_t} \frac{1}{\sqrt{2s_x}} \leq \prod_x \frac{1}{\sqrt{2s_x \wedge 1}} =: Q.$$

Combined with (4) we conclude that

$$M_t^{\sigma\Phi} \leq C(\Phi, |V|, W)Q.$$

Note that Q does not depend on t or T. On the other hand, by the independence of the s_x's, and the square-root together with the positive density of s_x at 0, we obtain that

$$EQ \leq \left(C(W) \int_0^\infty \frac{e^{-C(W)r}}{\sqrt{2r} \wedge 1} \right)^{|V|} < \infty,$$

completing the proof. $\square$