Elliptic functions

Let \( \omega_1, \omega_2 \) be two complex numbers such that \( \omega_1 \) and \( \omega_2 \) are linearly independent over \( \mathbb{R} \).

**Definition 1.** Meromorphic function \( f(z) \) is said to be **doubly periodic** if for any \( z \in \mathbb{C} \)

\[
f(z + \omega_1) = f(z + \omega_2) = f(z).
\]

Let \( \tau := \omega_1/\omega_2 \). Clearly \( \tau \in \mathbb{C} \) is not purely real. Moreover, changing \( \omega_1 \) to \(-\omega_1\), if necessary, we can assume that \( \text{Im}(\tau) > 0 \). Considering function \( F(z) := f(\omega_1 z) \) instead of \( f \), we will get a doubly-periodic function with the periods \( 1 \) and \( \tau \). Hence, from now on we assume that periods of \( f \) are \( 1 \) and \( \tau \), so that

\[
f(z + n + m\tau) = f(z), \quad \text{for all } n, m \in \mathbb{Z}.
\]

We say that \( 1 \) and \( \tau \) generate **lattice**

\[
\Lambda := \{ n + m\tau \mid n, m \in \mathbb{Z} \}.
\]

and call two points \( z, w \in \mathbb{C} \) equivalent modulo \( \Lambda \) if \( z - w \in \Lambda \).

Let \( \Pi \) be the parallelogram generated by vectors \( 1 \) and \( \tau \):

\[
\begin{array}{c}
\text{Im} \\
\downarrow \quad \tau \quad \downarrow \\
1 \\
\text{Re}
\end{array}
\]

This is **fundamental parallelogram**, i.e. any point \( z \in \mathbb{C} \) is equivalent to a unique point in \( \Pi \) (you have to include only one vertex and two adjacent sides into \( \Pi \) for it to work). Therefore any doubly periodic function \( f(z) \) is uniquely determined by its behavior on \( \Pi \).

**Remark 2.** The same will be true for any translate of \( \Pi \): a parallelogram of the form \( \Pi + h, h \in \mathbb{C} \).

The following theorem states that there is no interesting holomorphic doubly periodic functions.

**Theorem 3.** An **entire** doubly periodic function is constant.

**Proof.** Being continuous on a compact region \( \overline{\Pi} \), function \( |f(z)| \) is bounded. Therefore, by Liouville’s theorem, \( f(z) \) is constant.

Therefore, to get interesting doubly-periodic functions, we have to allow for poles. A non-constant doubly-periodic function is called **elliptic function**. Such function must have finitely many poles in any given bounded set. In particular, there are finitely many poles in \( \Pi \). It turns out, elliptic function \( f(z) \) has to have at least two poles (or a double pole) in \( \Pi \).

**Theorem 4.** The total number of poles of an elliptic function \( f(z) \) (counted with multiplicities) is \( \geq 2 \).

**Proof.** If there are no poles on \( \partial \Pi \), we can apply residue theorem:

\[
\int_{\partial \Pi} f(z)dz = 2\pi i \sum \text{res}(f).
\]

On the other hand, the integral on the left hand side is zero since \( f(z) \) is doubly periodic (and integrals over the opposite sides of \( \Pi \) cancel out). Hence sum of the residues is 0, so \( f(z) \) can not have only one simple pole in \( \Pi \).

If there are poles on \( \partial \Pi \), we can shift \( \Pi \) a bit by substituting it with \( \Pi + h \) for some \( h \in \mathbb{C} \).
**Definition 5.** Number of poles (with multiplicities) of an elliptic function \( f(z) \) is called its **order** (not to be confused with the order of a zero or a pole).

**Theorem 6.** Elliptic function \( f(z) \) of order \( m \) has exactly \( m \) zeros in \( \Pi \).

**Corollary 7.** Elliptic function of order \( m \) takes any value \( c \in \mathbb{C} \) exactly \( m \) times.

**Proof of the corollary.** Consider zeros of an elliptic function \( f(z) - c \).

**Proof of the theorem.** Assume first that \( f(z) \) does not have zeros or poles on \( \partial \Pi \). Then by argument principle

\[
\int_{\partial \Pi} \frac{f'(z)}{f(z)} \, dz = 2\pi i (N_z - N_p),
\]

where \( N_z \) and \( N_p \) are numbers of zeros and poles of \( f(z) \). By periodicity, the integral must vanish, so since \( N_p = m \), we must have \( N_z = 0 \).

Up to this point, it remains an open question if elliptic functions exist.

**Weierstrass \( \wp \)-function**

Before discussing construction of \( \wp \)-function, let us recall the partial fraction expansion of \( \pi \cot(\pi z) \). We knew that this function is \( \mathbb{Z} \)-periodic and that it has poles at all integers. It was natural to try to compare it to the function defined by the infinite sum

\[
\sum_{n \in \mathbb{Z}} \frac{1}{z + n}.
\]

The problem is that this sum does not converge, and we had to tweak it by considering instead the sum

\[
\frac{1}{z} + \sum_{n \in \mathbb{Z}, n \neq 0} \left( \frac{1}{z + n} - \frac{1}{n} \right).
\]

On every compact domain, the \( n \)-th term of this sum can be bounded by \( C/n^2 \), hence the sum is absolutely and uniformly convergent on compact subsets \( \mathbb{C} - \mathbb{Z} \).

The idea behind construction of \( \wp(z) \) is to mimic the above to make sense of the divergent infinite double sum

\[
\sum_{\omega \in \Lambda} \frac{1}{(z + \omega)^2}.
\]

We will follow a similar trick as for the infinite sum of \( \pi \cot(\pi z) \), and consider an infinite double sum

\[
\frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left( \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right),
\]

where \( \Lambda^* := \Lambda - \{(0, 0)\} \). We note that

For the term in the brackets and \( z \) bounded, we have

\[
\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} = \frac{-z^2 - 2z\omega}{(z + \omega)^2 \omega^2} = O(1/\omega^3), \quad \omega \to \infty.
\]

It is an easy exercise to verify at this point that the series in (1) is absolutely convergent. The function defined by this series is called **Weierstrass \( \wp \) function**:

\[
\wp(z) := \frac{1}{z^2} + \sum_{(n,m) \neq (0,0)} \left( \frac{1}{(z + n + mt)^2} - \frac{1}{(n + mt)^2} \right).
\]
Clearly \(\wp(z)\) is even. Due to the uniform convergence, function \(\wp(z)\) is meromorphic function with double poles at \(\Lambda\). For the same reason, derivative of \(\wp(z)\) can be computed by term-wise differentiation:

\[
\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z + \omega)^3}.
\]  

(2)

**Exercise 1.** Show that function \(\wp(z)\) is double periodic.

Hint: use the facts that \(\wp(z)\) is even and \(\wp'(z)\) is doubly periodic (due to the invariance of the sum (2) under shifts \(z \mapsto z + \omega\)).

**Properties of \(\wp(z)\)**

Since \(\wp(z)\) is even, and doubly periodic, it is easy to see that \(\wp'(z)\) must vanish at the ‘half-periods’ \(1/2, \tau/2\) and \((1 + \tau)/2\). Since \(\wp'(z)\) is of order 3, these must three simple zeros of \(\wp'(z)\). In particular, if we set

\[
\wp(1/2) = e_2, \quad \wp(\tau/2) = e_2, \quad \wp\left(\frac{1+\tau}{2}\right) = e_3,
\]

we see that each of the function \(\wp(z) - e_i\) has a double zero at the corresponding half-period. This implies that all the number \(e_1, e_2, e_3\) are distinct, since \(\wp(z) - c\) can have only two zeros with multiplicities for any given \(c\).

The following theorem provides the key property of the \(\wp\)-function.

**Theorem 8.** The function \((\wp')^2\) is the cubic polynomial in \(\wp(z)\):

\[
(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).
\]

**Proof.** The ratio of the left hand side and \((\wp - e_1)(\wp - e_2)(\wp - e_3)\) is a doubly periodic function with no zeros. Therefore, it must be a constant. Comparing the leading term of the Laurent series of \((\wp')^2\) and the right hand side we find, that the constant is \(4\).

In a certain sense, \(\wp\) and \(\wp'\) are universal elliptic functions.

**Theorem 9.** Every elliptic function \(f\) with periods 1 and \(\tau\) is a rational function of \(\wp\) and \(\wp'\).

**Proof.** Any elliptic function \(f(z)\) is a sum of an odd and an even elliptic function:

\[
f(z) = f_{\text{odd}}(z) + f_{\text{even}}(z) := \frac{f(z) - f(-z)}{2} \quad \frac{f(z) + f(-z)}{2}
\]

and \(f_{\text{odd}}/\wp'\) is also even. Therefore, it is enough to prove that any even elliptic function can be represented as a rational function of \(\wp\). The idea is to use \(\wp(z)\) to construct an elliptic function with prescribed zeros and poles.

Given any even elliptic \(F(z)\), if \(a \notin \Lambda\) is a zero of \(F\), then \(-a\) is also a zero, and \(a\) is equivalent to \(-a\) modulo \(\Lambda\) if and only if \(a\) is a half period. In the latter case zero at \(a\) has even multiplicity.

Dividing \(F\) by \(\wp(z) - \wp(a)\) we would kill these zeros at \(a\) and \(-a\). Repeating this procedure we would kill all the zero in \(\Pi\), introducing possibly zeros at points of \(\Lambda\).

Similarly we can kill all poles of \(F\) outside \(\Lambda\), by multiplying \(F\) with \(\wp(z) - \wp(b)\). We end with a function which does not have any zeros or poles in \(\mathbb{C} - \Lambda\). Such a function can not have a pole and a zero in \(\Pi\), therefore it must be constant.