Lecture 20

Conformal mappings

Given a differentiable map \( F : \mathbb{R}^2 \supset U \rightarrow \mathbb{R}^2 \) we have its induced action on vectors \( v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \) at \( x \in U \) via

\[
F_*(x)v = \begin{bmatrix}
\frac{\partial F_1(x)}{\partial x_1}v_1 + \frac{\partial F_1(x)}{\partial x_2}v_2 \\
\frac{\partial F_2(x)}{\partial x_1}v_1 + \frac{\partial F_2(x)}{\partial x_2}v_2
\end{bmatrix} = \text{Jac}(F(x)) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2,
\]

where \( \text{Jac}(F(x)) \) is the Jacobi matrix of \( F \) at \( x \).

One of the key features of holomorphic mappings \( f : U \rightarrow \mathbb{C} \) is \textit{conformality}. Namely, given a point \( z_0 \) such that \( f'(z_0) \neq 0 \), map \( f \) preserves angles between rooted vectors at \( z_0 \).

\textbf{Lemma 1.} Linear map \( A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) preserves oriented angles between vectors if and only if \( A \) is of the form

\[
A = \begin{bmatrix}
a & b \\
-b & a
\end{bmatrix},
\]

where \( a, b \in \mathbb{R} \) and \( a^2 + b^2 \neq 0 \).

\textit{Proof.} Exercise. \( \square \)

\textbf{Theorem 2.} Real differentiable map \( f : U \rightarrow \mathbb{C} \) has complex derivative \( f'(z_0) \neq 0 \) at \( z_0 \in U \) if and only if \( f \) preserves angles between vectors at \( z_0 \).

\textit{Proof.} By the Lemma, \( f \) preserves angles if and only if its real derivative is given by a matrix of the form

\[
\begin{bmatrix}
a & b \\
-b & a
\end{bmatrix},
\]

with \( a^2 + b^2 \). By Cauchy-Riemann identities a real differentiable map has differential of this form if and only if it is holomorphic. \( \square \)

\textbf{Definition 3.} A holomorphic function \( f : U \rightarrow \mathbb{C} \) is called a \textit{conformal map}, if its derivative does not vanish.

\textbf{Example 4.} Function \( f(z) = z^2 \) is a conformal mapping from \( \mathbb{C} - \{0\} \) onto \( \mathbb{C} - \{0\} \).

Fundamental question of complex analysis is to classify open subsets \( U \subset \mathbb{C} \) up to conformal equivalence. This raises two questions:

\textbf{Question.} Given \( U, V \subset \mathbb{C} \) does there exists a holomorphic bijection \( f : U \rightarrow V \) (such \( f \) is called conformal equivalence between \( U \) and \( V \))?

\textbf{Question.} Given \( U \subset \mathbb{C} \) what is the group of holomorphic automorphisms \( f : U \rightarrow U \)?

In general, these questions are difficult to answer. However, both of them have a remarkably simple answer if we additionally assume that \( U \) and \( V \) are simply connected. This is the content of the celebrated Riemann mapping theorem.
Riemann mapping theorem

**Theorem 5** (Riemann mapping theorem). Suppose that a connected open subset $U \subset \mathbb{C}$ is proper and simply-connected. Given $z_0 \in U$ there exists a unique bijective holomorphic function $F : U \rightarrow \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) \in \mathbb{R}_{>0}$.

**Proof of the uniqueness.** If $F_1$ and $F_2$ are two such functions, then $H := F_1 \circ F_2^{-1}$ is an automorphism of $\mathbb{D}$ fixing the origin. By a consequence of Schwarz lemma, any such automorphism has a form $H : z \mapsto e^{i\theta}z$. Conditions $F_1'(z_0), F_2'(z_0) \in \mathbb{R}_{>0}$ imply that $H'(0)$ is also real and positive, therefore $H(z) = z$. □

The existence part of the Riemann mapping theorem is one of the most important and fundamental theorems of our course. Surprisingly, for a result of such significance, this theorem has a rather easy proof.

Normal families

**Definition 6.** Consider an open connected subset $U \subset \mathbb{C}$ and a family $\mathcal{F}$ of complex-valued functions on $U$. We say that $\mathcal{F}$ is

- **normal** if for every sequence $\{f_i\}$ from $\mathcal{F}$ there is a subsequence $\{f_{k_i}\}$ converging on compact subsets of $U$.
- **locally uniformly bounded** if for any compact $K \subset U$ there exists $M > 0$ such that $|f(z)| < M$ for all $f \in \mathcal{F}$ and $z \in K$.
- **equicontinuous** on a compact set $K \subset U$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

  $$
  \text{for all } (z, w) \in K \text{ with } |z - w| < \delta \text{ and any } f \in \mathcal{F}
  \quad |f(z) - f(w)| < \varepsilon.
  $$

**Remark 7.** If there exists a constant $M > 0$ such that for every $f \in \mathcal{F}$ and $z \in K$ we have $|f'(z)| < M$, then $\mathcal{F}$ is equicontinuous on $K$.

**Theorem 8** (Montel’s theorem). Suppose $\mathcal{F}$ is a family of holomorphic functions on $U$ that is uniformly bounded on compact subsets of $U$. Then

1. $\mathcal{F}$ is equicontinuous on every compact subset $K \subset U$;
2. $\mathcal{F}$ is a normal family.

**Proof.** First, let us prove equicontinuity. This part relies in Cauchy’s theorem and essentially uses the fact that the functions in our family are holomorphic.

Given $K \subset U$ take $r > 0$ such that for every $z_0 \in K$ we have $B_{2r}(z_0) \subset U$. Let $N_{2r}(K)$ be a $2r$-neighbourhood of $K$:

$$
N_{2r}(K) = \{z \in \mathbb{C} \mid |z - w| \leq 2r \text{ for some } w \in K\}.
$$

By our assumption on $r$, $N_{2r}(K)$ is compact and contained in $U$. Hence, $\mathcal{F}$ is uniformly bounded by some constant $B > 0$ on $N_{2r}(K)$.

Given any $z, w \in K$ inside a disk $B_r(z_0)$ for $\gamma = \partial B_{2r}(z_0)$, we have

$$
f(z) - f(w) = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta.
$$

Now, for any $\zeta \in \gamma$

$$
\left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| = |z - w| \left| \frac{1}{(\zeta - z)(\zeta - w)} \right| \leq \frac{|z - w|}{r^2}.
$$
Therefore

$$|f(z) - f(w)| \leq B|z - w|.$$  

Since this inequality holds for all $f \in \mathcal{F}$ and $z, w \in K$, we conclude that $\mathcal{F}$ is equicontinuous on $K$.

Now we prove that $\mathcal{F}$ is normal under assumptions of equicontinuity and uniform boundedness. This a version of a general statement known as Arzelà–Ascoli theorem.

Let $\{f_n\}$ be a sequence from $\mathcal{F}$. Pick an everywhere dense sequence of points $\{w_j\} \subset U$, e.g., take all the points with rational coordinates. The sequence of values $\{f_n(w_j)\}_{n \in \mathbb{N}}$ is bounded, therefore we can choose a convergent subsequence $\{f_{n1}(w_1)\}_{n \in \mathbb{N}}$. At the next step we consider a bounded sequence $\{f_{n2}(w_2)\}$ and choose its convergent subsequence $\{f_{n3}(w_2)\}$. Repeating this process for all points $w_j$, we extract a diagonal sequence of functions

$$\{g_n\}_{n \in \mathbb{N}}, \quad g_n := f_{n, n}$$

such that the values at each of the points $w_j$ converge.

We claim that $\{g_n\}$ uniformly converges on $K$. Fix $\varepsilon > 0$. Since $\{g_n\}$ is uniformly equicontinuous on $K$, we can find the corresponding $\delta > 0$ and cover $K$ by a finite collection of balls $B_{\delta}(w_1), \ldots, B_{\delta}(w_k)$.

For $n, m > N(\varepsilon)$ large enough, and $z \in B_{\delta}(w_j)$ we have

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| + |g_m(w_j) - g_m(z)| < 3\varepsilon,$$

where the first and the last summands are bounded due to equicontinuity, and the middle term due to the fact that $\{g_n(w_j)\}_{n \in \mathbb{N}}$ converges for $j \in 1, \ldots, k$. 

\section*{Proof of the Riemann mapping}

\textbf{Lemma 9.} If $\{f_n\}$ is a sequence of injective holomorphic functions in $U \subset \mathbb{C}$ that converges uniformly on every compact subset $K \subset U$ to a holomorphic function $f(z)$, then $f(z)$ is either a constant or also injective.

\textbf{Proof.} Assume on the contrary that $f(z_1) = f(z_2)$ and $f$ is not constant. Then the function $g(z) := f(z) - f(z_1)$ has isolated zeros at $z_1$ and $z_2$, while $g_n(z) := g(z) - g(z_1)$ have isolated zero only at $z_1$.

By argument principle for a small circle $C_r(z_2)$ enclosing $z_2$, we have

$$\frac{1}{2\pi i} \int_{C_r(z_2)} \frac{g'(z)}{g(z)} \, dz = 1,$$

while

$$\frac{1}{2\pi i} \int_{C_r(z_2)} \frac{g_n'(z)}{g_n(z)} \, dz = 0.$$

This is a contradiction, since the integrands are uniformly convergent on $C_r(z_2)$. 

Now we turn onto the proof of the theorem.

\textbf{Step 1.} Suppose $U \subset \mathbb{C}$ is a proper simply-connected subset of $\mathbb{C}$. We claim that $U$ is conformally equivalent to an open subset of $\mathbb{D}$.

Indeed, pick $\alpha \notin U$. There exists a well defined function $f(z) = \sqrt{z - \alpha}$ in $U$. Clearly $f(z)$ does not take the same value twice, nor the opposite values, since $f(z)^2 = z - \alpha$.

Now, by open mapping, $f(U)$ contains some open disk $D = B_r(w)$, therefore it misses the open disk $B_r(-w)$. Therefore the map $F(z) = \frac{1}{f(z) + w}$ maps bijectively $U$ onto an open subset of $B_{1/r}(0)$.
**Step 2.** Composing with scalings and translations if necessary, by the first step, we may assume that $U \subset \mathbb{D}$ and $0 \in U$. So we have a non-empty family
\[ \mathcal{F} := \{ f : U \to \mathbb{D} \mid \text{holomorphic, injective and } f(0) = 0 \}. \]
Clearly $\mathcal{F}$ is uniformly bounded. Let
\[ s := \sup_{f \in \mathcal{F}} |f'(0)|. \]
This supremum is finite, since $|f'(0)|$ is bounded by Cauchy’s estimates. Choose a sequence $\{f_n\} \subset \mathcal{F}$ such that $|f_n'(0)| \to s$. By Montel’s theorem, sequence $\{f_n\}$ converges to a holomorphic function $f(z)$, moreover, by the above Lemma, since $f(z)$ is not constant\(^1\), we see that $f(z)$ is injective. Also, by continuity $|f(z)| \leq 1$ on $U$, so $f \in \mathcal{F}$ and $|f'(0)| = s$.

**Step 3.** Function $f(z)$ constructed above conformally maps $U$ onto $\mathbb{D}$.
Suppose, on the contrary that $\alpha \in \mathbb{D}$ does not belong to $f(U)$. Consider an automorphism of $\mathbb{D}$ interchanging $\alpha$ and $0$
\[ \psi_\alpha := \frac{\alpha - z}{1 - \bar{\alpha}z}. \]
Then simply connected region $W := (\psi_\alpha \circ f)(U)$ does not contain $0$ and we can define $g(w) := \sqrt{w}$ in $W$. Consider new function
\[ F := \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f. \]
It is easy to see that $F(0) = 0$, and $F : U \to \mathbb{D}$ is injective, so $F \in \mathcal{F}$. Then we have:
\[ f = \Phi \circ F, \]
where $\Phi : \mathbb{D} \to \mathbb{D}$ with $\Phi(0) = 0$ is non-injective. By Schwarz lemma we must have $|\Phi'(0)| < 1$, since otherwise $\Phi$ is an automorphism. Therefore
\[ |f'(0)| < |F'(0)| \]
which contradicts the maximality of $|f'(0)|$.
This proves that $f : U \to \mathbb{D}$ is not only injective but also surjective, and $f^{-1}$ is well-defined, since $f' \neq 0$ on $U$.

\[^1\text{By the way, why?}\]