PULLBACK OF A CONNECTION ON A VECTOR BUNDLE

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This note is motivated by the fact that the discussion of Jacobi fields and their covariant derivatives found in most textbooks is not rigorous, because the exponential map is not necessarily an immersion. It is necessary to pull back the tangent bundle of the manifold and the Levi-Civita connection using the parameterization of constant speed geodesics. However, it is difficult to find the definitions of the pullback of a vector bundle and the pullback of a connection on the vector bundle.

1. Pullback of a vector bundle

Let E be a rank k vector bundle over a smooth manifold M and $f : M \to N$ be a smooth map from M to a smooth manifold N.

We can define the pullback bundle f^*E as follows. As a set, f^*E is the disjoint union

$$f^*E = \sqcup_{x \in M} E_{f(x)}.$$

A (not necessarily smooth) section of f^*E is a map $t : M \to f^*E$, where $t(x) \in E_{f(x)}$. Choose a local smooth frame $(s_1, \dots, s_k)$ of E. The smooth vector bundle structure of f^*E is uniquely determined by specifying that $(f^*s_1, \dots, f^*s_K)$ is a local smooth frame of f^*E. In particular, a local section t of f^*E is defined to be smooth if and only if there exist smooth functions $a^1, \dots, a^k$ such that

$$t = a^i f^* s_i.$$

It is straightforward to show that this is equivalent to asserting that f^*s is a local smooth section of f^*E for any local smooth section s of E. This in turn shows that the definition of the smooth bundle structure of f^*E does not depend on the frame $(s_1, \dots, s_k)$ used.

In modern terms, the pullback bundle f^*E is defined using sheaves. Recall that a sheaf is the sheaf of local smooth sections of a smooth vector bundle if and only if it is locally free with respect to the local ring of smooth functions. The description above is the observation that the pullback of the sheaf of local smooth sections of E is a subsheaf of a unique locally free sheaf with respect to the local ring of smooth function on M.

2. Pullback of connection

The pullback of a connection ∇ on E is the connection $f^*\nabla$ on f^*E defined as follows.

The simple observation, which can be proved using a local frame of E, is that, if t is a local section of f^*E and $c : (-\delta, \delta) \to M$ is a smooth map, then $s = t \circ c$ defines a smooth map $s : (-\delta, \delta) \to E$. We can therefore define

$$f^*\nabla_V t(x) = (\nabla_{f_* V} s)(f(x)),$$

where $c'(0) = V$.

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