MOMENT-ENTROPY INEQUALITIES

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Abstract. It is shown that the product of the Rényi entropies of two independent random vectors provides a sharp lower bound for the expected value of the moments of the inner product of the random vectors. This new inequality contains important geometry (such as extensions of one of the fundamental affine isoperimetric inequalities, the Blaschke-Santaló inequality).

Introduction

Vitale (1996a, 1996b, 2001) presents evidence of a surprising connection between probability and analytic convex geometry. In this paper we contribute additional evidence of this unexpected link between these subjects.

It will be shown that the product of the Rényi entropies of two independent random vectors provides a sharp lower bound for the expected value of the moments of the inner product of the random vectors. Our new inequality encodes important geometry. For example, a non-technical version of the Blaschke-Santaló inequality for compact sets is but one special case.

This paper deals with random vectors in Euclidean n-space, $\mathbb{R}^n$. For vectors $x, y \in \mathbb{R}^n$ let $x \cdot y$ denote their inner product. If $\phi \in GL(n)$, then we use ϕ^{-t} to denote the inverse of the transpose of ϕ.

If X is a random vector in $\mathbb{R}^n$ with density f, then for $\lambda > 0$, the λ-Rényi entropy of X is defined (see e.g., Cover and Thomas (1992) and Gardner (2002)) by

$$\mathrm{Ent}_\lambda(X) = \begin{cases} \frac{1}{1-\lambda} \log \int_{\mathbb{R}^n} f(x)^\lambda \, dx & \lambda \neq 1, \\ -\int_{\mathbb{R}^n} f(x) \log f(x) \, dx & \lambda = 1. \end{cases}$$

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The λ-Rényi entropy power of X is defined as

$$N_\lambda(X) = e^{\mathrm{Ent}_\lambda(X)}, \qquad \text{with} \qquad N_\infty(X) = \lim_{\lambda\to\infty} N_\lambda(X).$$

A random vector in $\mathbb{R}^n$, with density function f, has *finite* p^{th} *moment* provided that

$$\int_{\mathbb{R}^n} |x|^p f(x)dx < \infty,$$

where $|x|$ denotes the ordinary Euclidean norm of $x \in \mathbb{R}^n$.

For $\phi \in GL(n)$, and $p, \lambda > 0$, define densities $\phi_{p,\lambda}\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\phi_{p,\lambda}(x) = \begin{cases} b\,(1+|\phi x|^p)^{\frac{1}{\lambda-1}} & \lambda < 1, \\ b\, e^{-|\phi x|^p} & \lambda = 1, \\ b\,(1-|\phi x|^p)_+^{\frac{1}{\lambda-1}} & \lambda > 1, \end{cases}$$

where $(z)_+ = \max\{0, z\}$, and in each case $b = b_{p,\lambda}$ is chosen so that $\phi_{p,\lambda}$ is a density.

We shall prove an extended version of:

Theorem. *Suppose real $p \geq 1$ and real $\lambda > \frac{n}{n+p}$ are fixed. If X and Y are independent random vectors in $\mathbb{R}^n$ that have finite p^{th} moment, then*

$$E(|X \cdot Y|^p) \geq c_1 [N_\lambda(X) N_\lambda(Y)]^{\frac{p}{n}},$$

where the best possible c_1 is given by

$$c_1 = \frac{2}{n\pi^{p+\frac{1}{2}}} \Gamma\Big(\frac{p+1}{2}\Big)\Gamma\Big(\frac{n+p}{2}\Big)^{-1}\Gamma\Big(\frac{n}{2}+1\Big)^{1+\frac{2p}{n}} c_0^2,$$

$$c_0^{-n/p} = \begin{cases} \frac{n}{p}\Big(1-\frac{n(1-\lambda)}{p\lambda}\Big)^{\frac{1}{\lambda-1}}\Big(\frac{p\lambda}{n(1-\lambda)}-1\Big)^{\frac{n}{p}} B\big(\frac{n}{p}, \frac{1}{1-\lambda}-\frac{n}{p}\big) & \lambda < 1 \\ \Big(\frac{pe}{n}\Big)^{\frac{n}{p}}\Gamma\big(\frac{n}{p}+1\big) & \lambda = 1 \\ \frac{n}{p}\Big(1+\frac{n(\lambda-1)}{p\lambda}\Big)^{\frac{1}{\lambda-1}}\Big(\frac{p\lambda}{n(\lambda-1)}+1\Big)^{\frac{n}{p}} B\big(\frac{n}{p}, \frac{\lambda}{\lambda-1}\big) & \lambda > 1. \end{cases}$$

Equality occurs if and only if X has density a.e. $\phi_{p,\lambda}$ and Y has density a.e. $(a\phi^{-t})_{p,\lambda}$, with $a > 0$.

If $K \subset \mathbb{R}^n$ is a compact set with volume (i.e., Lebesgue measure) $V(K)$ and X has density $\mathbf{1}_K/V(K)$, then trivially $N_\infty(X) = V(K)$. Now if $K, L \subset \mathbb{R}^n$ are compact and we let X, Y have densities $\mathbf{1}_K/V(K), \mathbf{1}_L/V(L)$ then letting $\lambda \to \infty$ and $p \to \infty$ in the theorem gives the following inequality.

Corollary. *If $K, L \subset \mathbb{R}^n$ are compact, then*

$$\omega_n^2 \max_{x \in K, y \in L} |x \cdot y|^n \geq V(K)V(L).$$

Here $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$ denotes the volume of the unit ball in $\mathbb{R}^n$. If K is an origin-symmetric convex body and L is the polar of K, then the above inequality is the classical Blaschke-Santaló inequality, with sharp constant. See e.g., the books of Gardner (1995), Leichtweiß (1998), Schneider (1993), and Thompson (1996) (and also the article of Bourgain and Milman (1987)) for references regarding the Blaschke-Santaló inequality.

0. Dual mixed volumes of random vectors

Each non-negative $\rho \in L_q(S^{n-1})$ defines a star-shaped set $K = K_\rho \subset \mathbb{R}^n$ by

$$K = \{ ru : 0 \leq r \leq \rho(u) \text{ with } u \in S^{n-1} \}.$$

The set K is called the L_q-*star* generated by ρ and the function ρ is called the radial function of K (and is often written as ρ_K to indicate its relationship to K). We will not distinguish between L_q-stars whose radial functions are a.e. equal.

It is convenient to extend the definition of the radial function from S^{n-1} to $\mathbb{R}^n \setminus \{0\}$ by making it homogeneous of degree -1. Thus, for an L_q-star $K \subset \mathbb{R}^n$, the radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$ can be defined by $\rho_K(x) = \max\{ r \geq 0 : rx \in K \}$. From this it follows immediately that if $\phi \in GL(n)$ then the radial function of the star $\phi K = \{\phi x \colon x \in K\}$ is given by $\rho_{\phi K}(x) = \rho_K(\phi^{-1}x)$ for all $x \neq 0$. From this, and the homogeneity (of degree -1) of the radial function, it follows immediately that E is an origin-centered ellipsoid of positive volume if and only if there exists a $\phi \in GL(n)$ such that $1/\rho_E(x) = |\phi x|$, for all $x \neq 0$.

Elements of the dual Brunn-Minkowski theory of L_q-stars were studied by Klain (1996, 1997). Other extensions for the dual Brunn-Minkowski theory have been considered by Gardner and Volčič (1994) and Gardner, Vedel Jensen, and Volčič (2002).

An L_q-star whose radial function is both positive and continuous is called a *star body.* A star body that is convex is called a *convex body.* Note that throughout, convex bodies are assumed to contain the origin in their interiors.

If K is a convex body in $\mathbb{R}^n$, then the polar, K^*, of K is defined by

$$K^* = \{x \in \mathbb{R}^n \colon x \cdot y \leq 1 \quad \text{for all } y \in K\}.$$

It follows immediately from this definition that if $\phi \in GL(n)$, then $(\phi K)^* = \phi^{-t} K^*$. From this we see that ρ_1, ρ_2 are radial functions of polar reciprocal origin-centered ellipsoids if and only if there exists a $\phi \in GL(n)$ such that

$$1/\rho_1(x) = |\phi x|, \qquad \text{and} \qquad 1/\rho_2(x) = |\phi^{-t} x|, \tag{0.1}$$

for all $x \neq 0$.

Suppose $p > 0$. Define the dual mixed volume $\tilde{V}_{-p}(K, L)$ of an L_{n+p}-star K and a star body L by:

$$\tilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n+p} \rho_L(u)^{-p}\, du = \frac{n+p}{n} \int_K \rho_L(x)^{-p}\, dx, \tag{0.2}$$

where in the left integral the integration is with respect to Lebesgue measure on S^{n-1} and in the right integral the integration is with respect to Lebesgue measure on $\mathbb{R}^n$. Obviously, for each star body L,

$$\tilde{V}_{-p}(L, L) = V(L). \tag{0.3}$$

From (0.2) and the Hölder inequality we see that if K is an L_{n+p}-star and L is a star body, then we have the dual mixed volume inequality:

$$\tilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p}, \tag{0.4}$$

with equality if and only if there exists a $c > 0$ such that, a.e., $\rho_K = c\rho_L$. Thus for the volume of each L_{n+p}-star, K, we have

$$V(K)^{\frac{n+p}{n}} = \mathrm{Inf}\{\tilde{V}_{-p}(K, Q) : Q \text{ is a star body with } V(Q) = 1\}. \tag{0.5}$$

If X is a random vector in $\mathbb{R}^n$, that has finite p^{th} moment and L is star body in $\mathbb{R}^n$, then define the dual mixed volume $\tilde{V}_{-p}(X, L)$ by

$$\tilde{V}_{-p}(X, L) = \frac{n+p}{n} \int_{\mathbb{R}^n} \rho_L(x)^{-p} f(x)\, dx, \tag{0.6}$$

where f is the density function X. It will be on occasion convenient to write $\tilde{V}_{-p}(f, L)$ rather than $\tilde{V}_{-p}(X, L)$.

For $p, \lambda > 0$, define $p_\lambda \colon \mathbb{R}^+ \to \mathbb{R}^+$ by

$$p_\lambda(s) = \begin{cases} (1 + s^p)^{\frac{1}{\lambda - 1}} & \lambda < 1, \\ e^{-s^p} & \lambda = 1, \\ (1 - s^p)_+^{\frac{1}{\lambda - 1}} & \lambda > 1. \end{cases}$$

We shall use:

Lemma 0.1. *Suppose K is a star body in $\mathbb{R}^n$. For real $a, p, \lambda > 0$, with $\lambda > n/(n+p)$,*

$$\int_{\mathbb{R}^n} \rho_K^{-p}(x) p_\lambda(a/\rho_K(x))\, dx = a^{-(n+p)} \alpha_1 V(K),$$

where

$$\alpha_1 = n\int_0^\infty s^{n+p-1}p_\lambda(s)\,ds = \begin{cases} \frac{n}{p}B(\frac{n+p}{p}, \frac{\lambda}{1-\lambda} - \frac{n}{p}) & \lambda < 1, \\ \frac{n}{p}\Gamma(\frac{n+p}{p}) & \lambda = 1, \\ \frac{n}{p}B(\frac{n+p}{p}, \frac{\lambda}{\lambda-1}) & \lambda > 1. \end{cases} \tag{0.1.1}$$

Proof. Rewrite the integral over $\mathbb{R}^n$ as an integral over $S^{n-1}\times(0,\infty)$

$$\int_{\mathbb{R}^n} \rho_K^{-p}(x)p_\lambda(a/\rho_K(x))\,dx = \int_{S^{n-1}}\int_0^\infty \rho_K^{-p}(ru)p_\lambda(a/\rho_K(ru))r^{n-1}drdu,$$

and observe that the inner integral is easily evaluated. Specifically, for fixed $u \in S^{n-1}$, make the change of variable $s = [a/\rho_K(u)]r = a/\rho_K(ru)$ and observe

$$\begin{aligned}\int_0^\infty \rho_K^{-p}(ru)p_\lambda(a/\rho_K(ru))r^{n-1}dr &= a^{-(n+p)}\rho_K^n(u)\int_0^\infty s^{n+p-1}p_\lambda(s)\,ds \\ &= \alpha_1 a^{-(n+p)}\tfrac{1}{n}\rho_K^n(u). \quad \square\end{aligned}$$

Thus, if K is a star body and b is chosen so that $bp_\lambda(a/\rho_K)$ is a probability density and $a > 0$, then

$$\tilde{V}_{-p}(bp_\lambda(a/\rho_K), K) = ba^{-(n+p)}\alpha_1 V(K).$$

We shall need:

Lemma 0.2. *Suppose K is a star body in $\mathbb{R}^n$. For $a, p, \lambda > 0$, with $\lambda > n/(n+p)$,*

$$\int_{\mathbb{R}^n} p_\lambda(a/\rho_K(x))\,dx = a^{-n}\alpha_2 V(K),$$

where

$$\alpha_2 = n\int_0^\infty s^{n-1}p_\lambda(s)\,ds = \begin{cases} \frac{n}{p}B(\frac{n}{p}, \frac{1}{1-\lambda} - \frac{n}{p}) & \lambda < 1, \\ \frac{n}{p}\Gamma(\frac{n}{p}) & \lambda = 1, \\ \frac{n}{p}B(\frac{n}{p}, \frac{\lambda}{\lambda-1}) & \lambda > 1. \end{cases} \tag{0.2.1}$$

The proof is similar to that of Lemma 0.1.

We shall also use:

Lemma 0.3. *Suppose K is a star body in $\mathbb{R}^n$. For $a, p, \lambda > 0$, with $\lambda > n/(n+p)$,*

$$\int_{\mathbb{R}^n} p_\lambda(a/\rho_K(x))^\lambda\,dx = a^{-n}\alpha_3 V(K),$$

where

$$\alpha_3 = n\int_0^\infty s^{n-1}p_\lambda(s)^\lambda\,ds = \begin{cases} \frac{n}{p}B(\frac{n}{p}, \frac{\lambda}{1-\lambda} - \frac{n}{p}) & \lambda < 1, \\ \frac{n}{p}\Gamma(\frac{n}{p}) & \lambda = 1, \\ \frac{n}{p}B(\frac{n}{p}, \frac{\lambda}{\lambda-1} + 1) & \lambda > 1. \end{cases} \tag{0.3.1}$$

The proof is similar to that of Lemma 0.1.

1. Constrained maximum Rényi entropy

The following lemma presents the solution to the problem of maximizing the λ-Rényi entropy when the value of the dual mixed volume of a random vector is fixed.

Lemma 1.1. *Suppose K is a star body in $\mathbb{R}^n$ and real $p, \lambda, c > 0$, with $\lambda > \frac{n}{n+p}$. Consider the problem of finding*

$$\max \ \mathrm{Ent}_\lambda(X),$$

subject to the constraint that X be a random vector in $\mathbb{R}^n$, with finite p^{th} moment, such that

$$\tilde{V}_{-p}(X, K) = c.$$

Then, the unique maximum is achieved by the random vector whose density function is a.e.

$$bp_\lambda(a/\rho_K),$$

where $b > 0$ is chosen so that $bp_\lambda(a/\rho_K)$ is a probability density and $a > 0$ is chosen so that

$$\tilde{V}_{-p}(bp_\lambda(a/\rho_K), K) = c.$$

Proof. Suppose f is a probability density on $\mathbb{R}^n$ such that $\tilde{V}_{-p}(f, K) = c$.

For the sake of notational simplicity let M denote

$$M = \frac{n}{n+p}\tilde{V}_{-p}(f, K) = \int_{\mathbb{R}^n} \rho_K^{-p} f \, dx,$$

and let

$$g_{\lambda,p} = bp_\lambda(a/\rho_K),$$

where $b > 0$ is chosen so that $bp_\lambda(a/\rho_K)$ is a probability density and $a > 0$ is chosen so that

$$M = b\int_{\mathbb{R}^n} \rho_K^{-p} p_\lambda(a/\rho_K)\, dx = \int_{\mathbb{R}^n} \rho_K^{-p} g_{\lambda,p}\, dx = \int_{\mathbb{R}^n} \rho_K^{-p} f\, dx. \tag{1.1.1}$$

(I) Case $\lambda = 1$.

From the fact that $g_{1,p} = be^{-a^p/\rho_K^p}$, and the fact that f and $g_{1,p}$ are probability densities, together with the last identity in (1.1.1), and the definition of Ent_1, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} f \log(g_{1,p}/f)\, dx &= -\int_{\mathbb{R}^n} f \log f\, dx + \int_{\mathbb{R}^n} f \log g_{1,p}\, dx \\
&= -\int_{\mathbb{R}^n} f \log f\, dx + \int_{\mathbb{R}^n} (\log b - a^p \rho_K^{-p}) f\, dx \\
&= -\int_{\mathbb{R}^n} f \log f\, dx + \int_{\mathbb{R}^n} (\log b - a^p \rho_K^{-p}) g_{1,p}\, dx \\
&= \mathrm{Ent}_1(f) - \mathrm{Ent}_1(g_{1,p}).
\end{aligned}$$

From the strict concavity of the log function, we see that

$$\int_{\mathbb{R}^n} f \log(g_{1,p}/f)\, dx \le \int_{\mathbb{R}^n} \log g_{1,p}\, dx \le \log \int_{\mathbb{R}^n} g_{1,p}\, dx = 0,$$

with equality if and only if a.e. $f = g_{1,p}$. Thus

$$\mathrm{Ent}_1(f) \le \mathrm{Ent}_1(g_{1,p}),$$

with equality if and only if a.e. $f = g_{1,p}$.

(II) *Case* $\lambda \neq 1$.

From the fact that $g_{\lambda,p}$ is a density function, Lemmas 0.2, 0.3, and 0.1, we have

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} g_{\lambda,p}\, dx = ba^{-n}\alpha_2 V(K) \\ &\int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda}\, dx = b^{\lambda} a^{-n}\alpha_3 V(K) \\ M &= \int_{\mathbb{R}^n} \rho_K^{-p} g_{\lambda,p}\, dx = ba^{-(n+p)}\alpha_1 V(K). \end{aligned}$$

Thus

$$b^{1-\lambda}\int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda}\, dx = \frac{\alpha_3}{\alpha_2}, \qquad \text{and} \qquad a^p M = \frac{\alpha_1}{\alpha_2}. \tag{1.1.2}$$

We now divide the case $\lambda \neq 1$ into two subcases, sub-case $\lambda < 1$ and sub-case $\lambda > 1$.

(II_1) *Sub-case* $\lambda < 1$.

The Hölder inequality and the fact that f and $g_{\lambda,p}$ are probability densities, shows that

$$\int_{\mathbb{R}^n} g_{\lambda,p}^{1-\lambda} f^{\lambda}\, dx \le \left(\int_{\mathbb{R}^n} g_{\lambda,p}\, dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} f\, dx\right)^{\lambda} = 1, \tag{1.1.3}$$

with equality if and only if a.e. $f = g_{\lambda,p}$.

From the definition of $g_{\lambda,p}$ and the definition of p_λ, for $\lambda < 1$, we see that $g_{\lambda,p}^{1-\lambda} = b^{1-\lambda} - a^p \rho_K^{-p} g_{\lambda,p}^{1-\lambda}$. From this and the Hölder inequality, again, we have:

$$\begin{aligned} \int_{\mathbb{R}^n} g_{\lambda,p}^{1-\lambda} f^{\lambda}\, dx &= b^{1-\lambda}\int_{\mathbb{R}^n} f^{\lambda}\, dx - \int_{\mathbb{R}^n} a^p \rho_K^{-p} g_{\lambda,p}^{1-\lambda} f^{\lambda} \\ &\ge b^{1-\lambda}\int_{\mathbb{R}^n} f^{\lambda}\, dx - a^p \left(\int_{\mathbb{R}^n} \rho_K^{-p} g_{\lambda,p}\, dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} \rho_K^{-p} f\, dx\right)^{\lambda} \\ &= b^{1-\lambda}\int_{\mathbb{R}^n} f^{\lambda}\, dx - a^p M. \end{aligned}$$

This together with (1.1.3) shows that

$$b^{1-\lambda}\int_{\mathbb{R}^n} f^{\lambda}\, dx \le a^p M + 1, \tag{1.1.4}$$

with equality if and only if a.e. $f = g_{\lambda,p}$.

In this sub-case (1.1.2), together with (0.1.1), (0.2.1) and (0.3.1) give:

$$b^{1-\lambda}\int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda}\,dx = \frac{\alpha_3}{\alpha_2} = \frac{\frac{\lambda}{1-\lambda}}{\frac{\lambda}{1-\lambda}-\frac{n}{p}}$$

$$a^p M = \frac{\alpha_1}{\alpha_2} = \frac{\frac{n}{p}}{\frac{\lambda}{1-\lambda}-\frac{n}{p}}.$$

It follows that

$$a^p M + 1 = b^{1-\lambda}\int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda}\,dx.$$

This and (1.1.4) gives the desired result that

$$\int_{\mathbb{R}^n} f^{\lambda}\,dx \le \int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda}\,dx,$$

with equality if and only if a.e. $f = g_{\lambda,p}$.

(II_2) *Sub-case* $\lambda > 1$.

From the the Hölder inequality we have

$$\int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda-1} f\,dx \le \left(\int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda}\,dx\right)^{1-1/\lambda}\left(\int_{\mathbb{R}^n} f^{\lambda}\,dx\right)^{1/\lambda}, \tag{1.1.5}$$

with equality if and only if a.e. $f = g_{\lambda,p}$.

From the definition of $g_{\lambda,p}$ and the definition of p_λ, for $\lambda > 1$, we see that in this case $g_{\lambda,p}^{\lambda-1} \ge b^{\lambda-1}(1 - a^p\rho_K^{-p})$. This and the fact that f is a probability density gives:

$$\int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda-1} f\,dx \ge b^{\lambda-1}\left(1 - a^p\int_{\mathbb{R}^n}\rho_K^{-p} f\,dx\right) = b^{\lambda-1}(1 - a^p M). \tag{1.1.6}$$

In this sub-case (1.1.2), together with (0.1.1), (0.2.1) and (0.3.1) give:

$$b^{1-\lambda}\int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda}\,dx = \frac{\alpha_3}{\alpha_2} = \frac{\frac{\lambda}{\lambda-1}}{\frac{\lambda}{\lambda-1}+\frac{n}{p}}$$

$$a^p M = \frac{\alpha_1}{\alpha_2} = \frac{\frac{n}{p}}{\frac{\lambda}{\lambda-1}+\frac{n}{p}}.$$

These identities yield

$$b^{\lambda-1}(1 - a^p M) = \int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda}\,dx,$$

and thus from (1.1.6) we have:

$$\int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda-1} f\,dx \ge \int_{\mathbb{R}^n} g_{\lambda,p}^{\lambda}\,dx.$$

This and (1.1.5) gives the desired result:

$$\int_{\mathbb{R}^n} f^{\lambda}\,dx \ge \int_{\mathbb{R}^n} g^{\lambda}\,dx,$$

with equality if and only if a.e. $f = g_{\lambda,p}$. $\square$

2. Inequalities between dual mixed volumes and Rényi entropy

Lemma 2.1. *Suppose real $p > 0$ and $\lambda > \frac{n}{n+p}$. If K is a star body in $\mathbb{R}^n$ and X is a random vector in $\mathbb{R}^n$ that has finite p^{th} moment, then*

$$\tilde{V}_{-p}(X, K) \geq \left(1 + \frac{p}{n}\right) c_0 [N_\lambda(X)/V(K)]^{\frac{p}{n}},$$

with equality if and only if X is a random vector with density function a.e. proportional to $p_\lambda(a/\rho_K)$, with some $a > 0$.

Proof. Abbreviate

$$M = \int_{\mathbb{R}^n} \rho_K^{-p} f\, dx = \frac{n}{n+p} \tilde{V}_{-p}(f, K).$$

Let

$$g_{\lambda,p} = b p_\lambda(a/\rho_K),$$

where b is chosen so that $bp_\lambda(a/\rho_K)$ is a probability density and a is chosen so that

$$M = b \int_{\mathbb{R}^n} \rho_K^{-p} p_\lambda(a/\rho_K)\, dx. \tag{2.1.1}$$

First note that from Lemma 1.1 we know that

$$N_\lambda(g_{\lambda,p}) \geq N_\lambda(f), \tag{2.1.2}$$

with equality if and only if a.e. $g_{\lambda,p} = f$.

From (2.1.1) and Lemma 0.1, we see that

$$M = ba^{-(n+p)} \alpha_1 V(K).$$

From the fact that $g_{\lambda,p}$ is a density function and and the fact that $g_{\lambda,p} = bp_\lambda(a/\rho_K)$, together with Lemma 0.2 we see that

$$1 = \int_{\mathbb{R}^n} g_{\lambda,p}\, dx = ba^{-n} \alpha_2 V(K).$$

Thus, we have:

$$a^p = \frac{\alpha_1}{\alpha_2 M} \tag{2.1.3}$$

$$b = \frac{a^n}{\alpha_2 V(K)}. \tag{2.1.4}$$

From Lemma 0.3, together with (2.1.3) and (2.1.4), we have

$$\int_{\mathbb{R}^n} g_{\lambda,p}^\lambda\, dx = b^\lambda a^{-n} \alpha_3 V(K) = \frac{1}{\alpha_2^\lambda} \left(\frac{\alpha_1}{\alpha_2 M}\right)^{\frac{n(\lambda-1)}{p}} \alpha_3 V(K)^{1-\lambda}.$$

Thus, for $\lambda \neq 1$,

$$N_\lambda(g_{\lambda,p})^{\frac{p}{n}} = \left(\frac{\alpha_3}{\alpha_2^\lambda}\right)^{\frac{p}{n(1-\lambda)}} \frac{\alpha_2}{\alpha_1} MV(K)^{\frac{p}{n}}. \tag{2.1.5}$$

Suppose $\lambda = 1$. From the definitions of $g_{1,p}$, p_1, Ent_1, and the fact that $bp_1(a/\rho_K)$ is a probability density together with Lemma 0.1, (2.1.4), and finally (0.1.1) together with (0.1.2) we have:

$$\begin{aligned}
\mathrm{Ent}_1(g_{1,p}) &= -\int_{\mathbb{R}^n} be^{-a^p\rho_K^{-p}}(\log b - a^p\rho_K^{-p})\,dx \\
&= -\log b\int_{\mathbb{R}^n} bp_1(a/\rho_K) + a^p b\int_{\mathbb{R}^n}\rho_K^{-p}p_1(a/\rho_K)\,dx \\
&= -\log b + ba^{-n}\alpha_1 V(K) \\
&= -\log b + \frac{\alpha_1}{\alpha_2} \\
&= -\log b + \frac{n}{p}.
\end{aligned}$$

This and the definition of N_1, together with (2.1.3) and (2.1.4), and finally (0.1.1) and (0.1.2) gives

$$N_1(g_{1,p})^{\frac{p}{n}} = eb^{-\frac{p}{n}} = \frac{ep}{n}\alpha_2^{\frac{p}{n}} MV(K)^{\frac{p}{n}}. \tag{2.1.6}$$

Therefore, from (2.1.2) and (2.1.5), (or from (2.1.2) and (2.1.6) when $\lambda = 1$), we have

$$M^{1/p} \geq c_0'[N_\lambda(f)/V(K)]^{1/n},$$

with equality if and only if $f = g_{\lambda,p}$, a.e., where c_0' is given by

$$c_0' = \begin{cases} \left(\frac{\alpha_3}{\alpha_2}\right)^{\frac{1}{n(\lambda-1)}} \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{p}} \alpha_2^{-\frac{1}{n}} & \lambda \neq 1 \\ \left(\frac{n}{ep}\right)^{\frac{1}{p}}\alpha_2^{-\frac{1}{n}} & \lambda = 1. \end{cases}$$

By using (0.1.1), (0.2.1), and (0.3.1) one now obtains the inequality of the lemma. □

A simple limit argument shows that the inequality of Lemma 2.1 holds when $\lambda = \infty$. However, in order to obtain the equality conditions we shall proceed in a different manner:

Lemma 2.2. *Suppose $p > 0$. If K is a star body in $\mathbb{R}^n$ and X is random vector in $\mathbb{R}^n$ that has finite p^{th} moment and bounded density, then*

$$\tilde{V}_{-p}(X,K)^{\frac{1}{p}} \geq [N_\infty(X)/V(K)]^{\frac{1}{n}},$$

with equality if and only if there exists an $a > 0$ such that the density function of X is, a.e., $\mathbf{1}_{aK}/V(aK)$.

Proof. Let f be the probability density of X, and let $\|f\|_\infty$ denote the essential supremum of f. We are assuming that $\|f\|_\infty < \infty$, and for convenience let $a = [\|f\|_\infty V(K)]^{-\frac{1}{n}}$.

From the definition of a radial function (and the fact that K is a star body) it follows immediately that $x \in \operatorname{int} aK$ if and only if $a > \rho_K^{-1}(x)$ or equivalently $\operatorname{int} aK$ is precisely the set on which the function $(a^p - \rho_K^{-p})_+$ is positive. This observation and the fact that f is a density function, together with definition (0.6), shows that

$$\begin{aligned}\int_{\mathbb{R}^n} (a^p - \rho_K^{-p})_+ f\,dx &\geq \int_{\mathbb{R}^n} (a^p - \rho_K^{-p}) f\,dx \\ &= a^p - \frac{n}{n+p}\tilde{V}_{-p}(f, K),\end{aligned}$$

with equality if and only if $f(x) = 0$ for almost all $x \notin \operatorname{int} aK$.

Again from the fact that $\operatorname{int} aK$ is precisely the set on which the function $(a^p - \rho_K^{-p})_+$ is positive, together with (0.2), the fact that $\tilde{V}_{-p}(\cdot, K)$ is homogeneous of degree $n+p$, and (0.3) we have:

$$\begin{aligned}\int_{\mathbb{R}^n} (a^p - \rho_K^{-p})_+ f\,dx &\leq \|f\|_\infty \int_{\mathbb{R}^n} (a^p - \rho_K^{-p})_+\,dx \\ &= \|f\|_\infty \int_{aK} (a^p - \rho_K^{-p})\,dx \\ &= \|f\|_\infty a^p \int_{aK} 1\,dx - \frac{n\|f\|_\infty}{n+p}\tilde{V}_{-p}(aK, K) \\ &= \frac{p}{n+p} a^{n+p}\|f\|_\infty V(K),\end{aligned}$$

with equality if and only if f is a.e. constant on $\operatorname{int} aK$.

Combining the above inequalities, and recalling that $a = [\|f\|_\infty V(K)]^{-\frac{1}{n}}$, gives

$$\begin{aligned}\tilde{V}_{-p}(f, K) &\geq \frac{n+p}{n} a^p - \frac{p}{n} a^{n+p}\|f\|_\infty V(K) \\ &= [\,\|f\|_\infty V(K)\,]^{-\frac{p}{n}},\end{aligned}$$

with equality if and only if f is a.e. constant on $\operatorname{int} aK$ and 0 on its complement.

This, and the fact that $N_\infty(f) = 1/\|f\|_\infty$, gives the desired inequality and the fact that equality holds if and only if, a.e., $f = \mathbf{1}_{aK}/V(aK)$. $\square$

3. A star body associated with a random vector

Suppose $p > 0$, and $f\colon \mathbb{R}^n \to \mathbb{R}$ is a density function that has finite p^{th} moment. Define the Borel measure μ_f on S^{n-1} by letting

$$\int_{S^{n-1}} q(u)\,d\mu_f(u) = \int_{\mathbb{R}^n} f(x) q(x/|x|)|x|^p\,dx,$$

for each $q \in C(S^{n-1})$. Since the measure μ_f is absolutely continuous with respect spherical Lebesgue measure, there is an essentially unique function $\bar{f} \in L_1(S^{n-1})$ such that $\bar{f} \geq 0$ and

$$\frac{1}{n}\int_{S^{n-1}} q(u)\bar{f}(u)\,du = \int_{\mathbb{R}^n} f(x) q(x/|x|)|x|^p\,dx,$$

for each $q \in C(S^{n-1})$. Thus, from f we get a unique L_{n+p}-star, T_pf, defined by $\rho_{T_pf}^{n+p} = \bar{f}$ such that

$$\tilde{V}_{-p}(T_pf, Q) = \int_{\mathbb{R}^n} f(x)\rho_Q(x)^{-p}dx, \tag{3.1}$$

for each star body Q. Note that $0 < V(T_pf) < \infty$ (since $V(T_pf) = \frac{1}{n}\int_{S^{n-1}} \bar{f}^{\frac{n}{n+p}}$), and define $S_pf = V(T_pf)^{\frac{1}{p}}T_pf$. The homogeneity (of degree n) of volume now immediately gives $V(S_pf)^{\frac{1}{n+p}} = V(T_pf)^{\frac{1}{p}}$. But $V(S_pf)^{-\frac{1}{n+p}}S_pf = T_pf$ and the fact that $\tilde{V}_{-p}(\cdot, Q)$ is homogeneous of degree $n+p$, lets us rewrite (3.1) as

$$\tilde{V}_{-p}(S_pf, Q)/V(S_pf) = \int_{\mathbb{R}^n} f(x)\rho_Q(x)^{-p}dx, \tag{3.2}$$

or by (0.2) equivalently

$$\frac{n+p}{nV(S_pf)}\int_{S_pf} \rho_Q(x)^{-p}dx = \int_{\mathbb{R}^n} f(x)\rho_Q(x)^{-p}dx, \tag{3.3}$$

for each star body Q.

If X random vector in $\mathbb{R}^n$ with density function f, that has finite p^{th} moment, then we will often write S_pX rather than S_pf. Thus, from (3.2) and (0.6) we see that S_pX can be defined by simply requiring that

$$\tilde{V}_{-p}(S_pX, Q)/V(S_pX) = \left(1 + \frac{p}{n}\right)\tilde{V}_{-p}(X, Q), \tag{3.4}$$

hold for each star body Q.

Observe that if $f \in C_c^\infty(\mathbb{R}^n)$ is positive in a neighborhood of the origin, then the L_{n+p} star, S_pf, is in fact a star body, and it follows that from (0.2) together with (3.2) that its radial function ρ_{S_pX} is given by

$$\frac{1}{V(S_pX)}\rho_{S_pX}(u)^{n+p} = n\int_0^\infty f(ru)r^{n+p-1}\,dr.$$

If $f \in C_c^\infty(\mathbb{R}^n)$ is positive in a neighborhood of the origin, then by taking $Q = S_pX$ in (3.4) and recalling (0.3) we see that

$$\tilde{V}_{-p}(X, S_pX) = 1 + \frac{p}{n}. \tag{3.6}$$

As will now be shown, the volume of the star associated with a random vector can be bounded from below by the λ-Rényi entropy power of the random vector. Although the inequality of our next lemma is stated without equality conditions, it is sharp.

Lemma 3.1. *Suppose $p > 0$ and $\lambda > \frac{n}{n+p}$. If X is a random vector in $\mathbb{R}^n$ that has finite p^{th} moment, then*

$$V(S_pX) \geq c_0^{n/p} N_\lambda(X).$$

Proof. Let f denote the density function of X. First note that if $f \in C_c^\infty(\mathbb{R}^n)$ is positive in a neighborhood of the origin, then from (3.6) and Lemma 2.1,

$$\left(1 + \frac{p}{n}\right)^{\frac{n}{p}} = \tilde{V}_{-p}(X, S_pX)^{\frac{n}{p}} \geq \left(1 + \frac{p}{n}\right)^{n/p} c_0^{n/p} N_\lambda(X)/V(S_pX). \tag{3.1.1}$$

This establishes the desired inequality for the special case where $f \in C_c^\infty(\mathbb{R}^n)$ is positive in a neighborhood of the origin.

To handle case for arbitrary f when $\lambda \neq 1$, choose a sequence of probability density functions $f_i \in C_c^\infty(\mathbb{R}^n)$ that are positive in a neighborhood of the origin and such that

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} f_i(x)\rho_Q(x)^{-p}dx = \int_{\mathbb{R}^n} f(x)\rho_Q(x)^{-p}dx, \tag{3.1.2}$$

for each star body Q, and

$$\limsup_{i\to\infty} N_\lambda(f_i) \geq N_\lambda(f). \tag{3.1.3}$$

Now suppose Q is a star body such that $V(Q) = 1$. From (3.1.2) and (3.2), followed by the dual mixed volume inequality (0.4), then (3.1.1), and finally (3.1.3), we have

$$\begin{aligned} \tilde{V}_{-p}(S_pf, Q)/V(S_pf) &= \lim_{i\to\infty} \tilde{V}_{-p}(S_pf_i, Q)/V(S_pf_i) \\ &\geq \limsup_{i\to\infty} V(S_pf_i)^{\frac{p}{n}} \\ &\geq c_0 \limsup_{i\to\infty} N_\lambda(f_i)^{\frac{p}{n}} \\ &\geq c_0 N_\lambda(f)^{\frac{p}{n}}. \end{aligned}$$

This together with (0.5) now completes the proof for the case of arbitrary f when $\lambda \neq 1$. The case of arbitrary f when $\lambda = 1$ now follows from the case $\lambda \neq 1$ by taking a simple limit. □

It will be convenient to re-define c_0 so that it is defined not only for positive λ but for $\lambda = \infty$ as well. To this end, define

$$c_0^{-n/p} = \begin{cases} \frac{n}{p}\left(1 - \frac{n(1-\lambda)}{p\lambda}\right)^{\frac{1}{\lambda-1}} \left(\frac{p\lambda}{n(1-\lambda)} - 1\right)^{\frac{n}{p}} B\left(\frac{n}{p}, \frac{1}{1-\lambda} - \frac{n}{p}\right) & \lambda < 1, \\ \left(\frac{pe}{n}\right)^{\frac{n}{p}} \Gamma\left(\frac{n}{p} + 1\right) & \lambda = 1, \\ \frac{n}{p}\left(1 + \frac{n(\lambda-1)}{p\lambda}\right)^{\frac{1}{\lambda-1}} \left(\frac{p\lambda}{n(\lambda-1)} + 1\right)^{\frac{n}{p}} B\left(\frac{n}{p}, \frac{\lambda}{\lambda-1}\right) & \lambda > 1, \\ \left(1 + \frac{p}{n}\right)^{\frac{n}{p}} & \lambda = \infty. \end{cases}$$

Taking $\lambda \to \infty$ in Lemma 3.1 (and noting that $\lim_{\lambda\to\infty} c_0 = (1 + p/n)^{-1}$) gives:

Lemma 3.2. *Suppose real $p > 0$ and X is a random vector in $\mathbb{R}^n$ that has finite p^{th} moment, then*

$$V(S_pX) \geq c_0^{n/p} N_\infty(X).$$

4. Moments of random variables

Suppose $p \geq 1$. If $K \subset \mathbb{R}^n$ is an L_{n+p}-star that is of positive volume, then its polar L_p-centroid body, $\Gamma_p^* K$, is the convex body whose radial function is given by

$$\rho_{\Gamma_p^* K}(u)^{-p} = \frac{1}{V(K)} \int_K |u \cdot x|^p dx. \tag{4.1}$$

From this definition, it is easily seen that if E is an ellipsoid centered at the origin then

$$\Gamma_p^* E \quad \text{is a dilate of } E^*. \tag{4.2}$$

Definitions (0.2) and (4.1) and Fubini's theorem show that for positive-volume L_{n+p}-stars $K, L \subset \mathbb{R}^n$ we have

$$\tilde{V}_{-p}(K, \Gamma_p^* L)/V(K) = \tilde{V}_{-p}(L, \Gamma_p^* K)/V(L). \tag{4.3}$$

Take $L = \Gamma_p^* K$ in (4.3), and from (0.3) get: If $K \subset \mathbb{R}^n$ is an L_{n+p}-star, then

$$V(K) = \tilde{V}_{-p}(K, \Gamma_p^* \Gamma_p^* K). \tag{4.4}$$

For $p = 1$ and a more restricted class of bodies identity (4.3) was first given in Lutwak (1990).

Lemma 4.2. *Let X and Y be independent random variables in $\mathbb{R}^n$ that have finite p^{th} moment, then for $p \geq 1$*

$$\tilde{V}_{-p}(X, \Gamma_p^* S_p Y) = E(|X \cdot Y|^p) = \tilde{V}_{-p}(Y, \Gamma_p^* S_p X).$$

Proof. Let f, g be the density functions of X, Y.

First note that an obvious limit argument in (3.3) shows that for each $x \in \mathbb{R}^n$,

$$\frac{n+p}{nV(S_p g)} \int_{S_p g} |x \cdot y|^p dy = \int_{\mathbb{R}^n} g(y) |x \cdot y|^p dy. \tag{4.2.1}$$

By using (0.6), (4.1), and (4.2.1), we have

$$\begin{aligned}
\tilde{V}_{-p}(X, \Gamma_p^* S_p Y) &= (1 + \tfrac{p}{n}) \int_{\mathbb{R}^n} \rho_{\Gamma_p^* S_p Y}^{-p}(x) f(x) dx \\
&= \frac{n+p}{nV(S_p Y)} \int_{\mathbb{R}^n} f(x) \int_{S_p Y} |x \cdot y|^p \, dy \, dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x \cdot y|^p f(x) g(y) \, dx \, dy \\
&= E(|X \cdot Y|^p).
\end{aligned}$$

To complete the proof observe that $E(|X \cdot Y|^p)$ is symmetric in X and Y. $\square$

Define c_2 by

$$c_2 = \left(\frac{\pi^{\frac{1}{2}+p} \left(1+\frac{p}{n}\right) \Gamma\left(\frac{p+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(1+\frac{n}{2}\right)^{2p/n}} \right)^{\frac{n}{p}}.$$

We shall use the following result of Lutwak and Zhang (1997) (see also Lutwak, Yang, and Zhang (2000) as well as Campi and Gronchi (2002)): If K is an origin-symmetric convex body, then

$$V(\Gamma_p^* K) V(K) \leq c_2, \tag{4.5}$$

with equality if and only if K is an ellipsoid centered at the origin.

Theorem 4.3. *Suppose real $p \geq 1$ and $\lambda \in (\frac{n}{n+p}, \infty]$. Let X and Y be independent random variables in $\mathbb{R}^n$ that have finite p^{th} moment (and are also of bounded density if $\lambda = \infty$), then*

$$E(|X \cdot Y|^p) \geq c_0^2 c_2^{-\frac{p}{n}} (1 + \tfrac{p}{n}) [N_\lambda(X) N_\lambda(Y)]^{\frac{p}{n}},$$

with equality for $\lambda < \infty$ if and only if there exists an $\phi \in GL(n)$ with X having density a.e. $\phi_{p,\lambda}$ and Y having density a.e. $(a\phi^{-t})_{p,\lambda}$, with $a > 0$, and equality for $\lambda = \infty$ if and only if the densities of X and Y are a.e. proportional to the characteristic functions of dilates of polar-reciprocal origin-centered ellipsoids.

Proof. With $i \neq j$, let $\{X_i, X_j\} = \{X, Y\}$ and let f_i, f_j denote the density functions of X_i, X_j. From Lemma 4.2, (3.4), the dual mixed volume inequality (0.4), Lemma 3.2, (4.5), (0.4), (4.4), and Lemma 3.2, we have

$$\begin{aligned} E(|X \cdot Y|^p) &= \tilde{V}_{-p}(X_i, \Gamma_p^* S_p X_j) \\ &= (1 + \tfrac{p}{n}) \tilde{V}_{-p}(S_p X_i, \Gamma_p^* S_p X_j)/V(S_p X_i) \\ &\geq (1 + \tfrac{p}{n}) [V(S_p X_i)/V(\Gamma_p^* S_p X_j)]^{\frac{p}{n}} \\ &\geq (1 + \tfrac{p}{n}) c_0 [N_\lambda(X_i)/V(\Gamma_p^* S_p X_j)]^{\frac{p}{n}} \\ &\geq (1 + \tfrac{p}{n}) c_0 c_2^{-\frac{p}{n}} N_\lambda(X_i)^{\frac{p}{n}} V(\Gamma_p^* \Gamma_p^* S_p X_j)^{\frac{p}{n}} \\ &\geq (1 + \tfrac{p}{n}) c_0 c_2^{-\frac{p}{n}} N_\lambda(X_i)^{\frac{p}{n}} \tilde{V}_{-p}(S_p X_j, \Gamma_p^* \Gamma_p^* S_p X_j)^{-1} V(S_p X_j)^{\frac{n+p}{n}} \\ &= (1 + \tfrac{p}{n}) c_0 c_2^{-\frac{p}{n}} N_\lambda(X_i)^{\frac{p}{n}} V(S_p X_j)^{\frac{p}{n}} \\ &\geq (1 + \tfrac{p}{n}) c_0^2 p c_2^{-\frac{p}{n}} N_\lambda(X_i)^{\frac{p}{n}} N_\lambda(X_j)^{\frac{p}{n}}. \end{aligned}$$

Suppose there is equality in the inequality of the theorem. Obviously, this implies equality in all of the inequalities above. Equality in the first inequality, by the equality conditions of the dual Minkowski inequality (0.4), implies that there exist $d_i > 0$ such that, a.e.,

$$\rho_{S_p X_i} = d_i \rho_{\Gamma_p^* S_p X_j}. \tag{4.3.1}$$

The fact that $\tilde{V}_{-p}(X_i, \Gamma_p^* S_p X_j) = (1+\frac{p}{n})c_0[N_\lambda(X_i)/V(\Gamma_p^* S_p X_j)]^{\frac{p}{n}}$ and the equality conditions of Lemma 2.1 (or Lemma 2.2 if $\lambda = \infty$) shows that there exist $a_i, b_i > 0$ such that, a.e.,

$$f_i = \begin{cases} b_i p_\lambda(a_i/\rho_{\Gamma_p^* S_p X_j}) & \lambda < \infty, \\ \mathbf{1}_{a_i \Gamma_p^* S_p X_j}/V(a_i \Gamma_p^* S_p X_j) & \lambda = \infty. \end{cases} \tag{4.3.2}$$

From the equality conditions of (4.5), we see that equality in the third inequality implies, that there exits an origin-centered ellipsoid, E_j, such that

$$\Gamma_p^* S_p X_j = E_j. \tag{4.3.3}$$

But (4.3.3) and (4.3.2) show that a.e.,

$$f_i = \begin{cases} b_i p_\lambda(a_i/\rho_{E_j}) & \lambda < \infty, \\ \mathbf{1}_{a_i E_j}/V(a_i E_j) & \lambda = \infty. \end{cases} \tag{4.3.4}$$

But (4.3.1) and (4.3.3) show that $\rho_{S_p X_i} = d_i \rho_{E_j}$, a.e., and hence by (4.1)

$$\Gamma_p^* S_p X_i = d_i' E_j^*, \tag{4.3.5}$$

for some $d_i' > 0$.

Note that (4.3.2) shows that, a.e.,

$$f_j = \begin{cases} b_j p_\lambda(a_j/\rho_{\Gamma_p^* S_p X_i}) & \lambda < \infty, \\ \mathbf{1}_{a_j \Gamma_p^* S_p X_i}/V(a_j \Gamma_p^* S_p X_i) & \lambda = \infty, \end{cases}$$

and when combined with (4.3.5) we see that, a.e.,

$$f_j = \begin{cases} b_j p_\lambda(a_j/\rho_{d_i' E_j^*}) & \lambda < \infty, \\ \mathbf{1}_{a_j' E_j^*}/V(a_j' E_j^*) & \lambda = \infty, \end{cases} \tag{4.3.6}$$

for some $a_j' > 0$. When (4.3.4) and (4.3.6) are combined with (0.1), we get the desired equality conditions. □

A comment regarding the above proof: Two extra steps were needed in the proof of the inequality of Theorem 4.3 because inequality (4.5) had not been established for the class of L_{n+p}-stars. The *class reduction* technique that is used to overcome this difficulty was introduced in Lutwak (1986).

Suppose $K, L \subset \mathbb{R}^n$ are compact. In Theorem 4.3, with $\lambda = \infty$, let X and Y be random vectors with density $\mathbf{1}_K/V(K)$ and $\mathbf{1}_L/V(L)$ and get:

Corollary 4.4. *If K and L are compact sets in $\mathbb{R}^n$, then for real $p \geq 1$,*

$$\int_K \int_L |x \cdot y|^p \, dxdy \geq \frac{n}{n+p} c_2^{-\frac{p}{n}} \left[V(K)V(L)\right]^{\frac{n+p}{n}},$$

with equality if and only if K and L are, up to sets of measure 0, dilates of polar-reciprocal, origin-centered ellipsoids.

The limiting case $p \to \infty$ of Corollary 4.4 gives:

Corollary 4.5. *If K and L are compact sets in $\mathbb{R}^n$, then*

$$\max_{x\in K, y\in L} |x\cdot y| \geq \omega_n^{-\frac{2}{n}} [V(K)V(L)]^{\frac{1}{n}}.$$

If K is an origin-symmetric convex body and L is its polar, then the above inequality is the classical Blaschke-Santaló inequality (and $\omega_n^{-2/n}$ is the precise constant that makes the inequality sharp).

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