

HOLONOMY IS CURVATURE

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Let E be a vector bundle over a smooth manifold M and ∇ a connection on E . The curvature of the connection is the section Ω of $\wedge^2 T^*M \otimes \text{Aut}(E)$ such that

$$(1) \quad \Omega(X, Y)e = ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]})e \in E_x,$$

for any $x \in M$, $X, Y \in T_x M$, $e \in E_x$.

Given a smooth curve $c : [0, 1] \rightarrow M$, the parallel transport of $e \in E_{c(0)}$ along c is defined to be the section $f : [0, 1] \rightarrow E$ such that the following hold for each $t \in [0, 1]$:

$$\begin{aligned} f(t) &\in E_{c(t)} \\ f(0) &= e \\ \nabla_T f(t) &= 0, \end{aligned}$$

where $T = \partial_t$. Denote $P_c e = f(1)$.

Let $c : [0, 1] \rightarrow M$ be a C^1 null-homotopic curve based at x . There exists a C^1 map $C : [0, 1] \times [0, 1] \rightarrow M$ satisfying the following for each $0 \leq s, t \leq 1$:

$$\begin{aligned} C(0, t) &= x \\ C(1, t) &= c(t) \\ C(s, 0) &= x \\ C(s, 1) &= x \end{aligned}$$

Given $e_x \in E_x$, let $e : [0, 1] \times [0, 1] \rightarrow E$ be C^2 section of C^*E satisfying the following for all $0 \leq s, t \leq 1$:

$$\begin{aligned} e(s, t) &\in E_{C(s, t)} \\ e(s, 0) &= e_x \\ \nabla_T e(1, t) &= 0 \\ \nabla_S e(s, t) &= 0, \end{aligned}$$

where $S = \partial_s$ and $T = \partial_t$. In particular,

$$e(s, 1) = P_c e_x.$$

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The approach shown here is inspired by the writings of Eschenburg, Heintze, Jost, Karcher.

Let E^* be the dual vector bundle of E . Given $\varepsilon_x \in E_x^*$, let $\varepsilon : [0, 1] \times [0, 1] \rightarrow E^*$ satisfy the following for all $0 \leq s, t \leq 1$:

$$\begin{aligned}\varepsilon(s, t) &\in E_{C(s,t)}^* \\ \varepsilon(0, t) &= \varepsilon_x \\ \varepsilon(s, 0) &= \varepsilon_x \\ \varepsilon(s, 1) &= \varepsilon_x \\ \nabla_S \varepsilon(s, t) &= 0.\end{aligned}$$

It follows that

$$\nabla_T \varepsilon(0, t) = 0.$$

Lemma 1.

$$\langle \varepsilon_x, P_c e_x - e_x \rangle = \int_{[0,1] \times [0,1]} \langle \varepsilon(s, t), C^* \Omega \rangle.$$

Proof.

$$\begin{aligned}\langle \varepsilon_x, P_c e_x - e_x \rangle &= \langle \varepsilon(0, 1), e(0, 1) \rangle - \langle \varepsilon(0, 0), e(0, 0) \rangle \\ &= \int_{t=0}^{t=1} \partial_t (\langle \varepsilon(0, t), e(0, t) \rangle) dt \\ &= \int_{t=0}^{t=1} \langle \varepsilon, \nabla_T e(0, t) \rangle dt \\ &= \int_{t=0}^{t=1} \left[\langle \varepsilon, \nabla_T e(1, t) \rangle - \int_{s=0}^{s=1} \partial_s (\langle \varepsilon, \nabla_T e(s, t) \rangle) \right] ds dt \\ &= - \int_{t=0}^{t=1} \int_{s=0}^{s=1} \langle \varepsilon, \nabla_S \nabla_T e(s, t) \rangle ds dt \\ &= \int_{t=0}^{t=1} \int_{s=0}^{s=1} \langle \varepsilon, \Omega(C_* T, C_* S) e(s, t) \rangle ds dt \\ &= \int_{[0,1] \times [0,1]} \langle \varepsilon, C^* \Omega \rangle e.\end{aligned}$$

□

A corollary of this is the Ambrose-Singer theorem [1]. An elegant presentation of the above can be found in [lecture notes of Werner Ballman](#).

REFERENCES

- [1] W. Ambrose and I. M. Singer. *A theorem on holonomy*. Trans. Amer. Math. Soc. 75 (1953), pp. 428–443. DOI: [10.2307/1990721](https://doi.org/10.2307/1990721).