DeTurck's trick for the prescribed Ricci equation

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Given a symmetric tensor $R = R_{ij}\, dx^i\, dx^j$, we want to solve for a Riemannian metric $\hat{g}$ such that

$$\mathrm{Ricci}(\hat{g}) = R. \tag{1}$$

DeTurck's trick is easier to explain, if we start with an approximate solution g to (1) and show how to deform it into a solution $\hat{g} = g + h$.

In other words, given a symmetric tensor $R = R_{ij}\, dx^i\, dx^j$ and a Riemannian metric $g = g_{ij}\, dx^i dx^j$, solve for a symmetric tensor $h = h_{ij}\, dx^i\, dx^j$ such that

$$\mathrm{Ricci}(g + h) = R. \tag{2}$$

This is equivalent to solving a system of PDE's given by

$$\frac{1}{2}(-g^{pq}\partial^2_{pq}h_{ij} + \partial^2_{ip}h_{jq} + \partial^2_{jp}h_{iq} - \partial^2_{ij}h_{pq}) + \text{ lower order terms in } h = (R - \mathrm{Ricci}(g))_{pq}, \tag{3}$$

which is a degenerate system of PDE's and therefore hard to solve directly. The degeneracy is due to the equivariance of (1) under the action of the diffeomorphism group. It is an example of what physicists call *gauge invariance*. This phenomenon was observed by physicists in at least two different settings: Gauge theories such as the Yang-Mills equations and the Einstein equation in general relativity. They discovered a trick, called *gauge breaking* or gauge fixing, which eliminates the degeneracy. DeTurck's trick can be viewed as an example of this.

The first step in DeTurck's trick is, instead of solving (2), solve for h and a diffeomorphism Φ satisfying

$$\mathrm{Ricci}(g + h) = \Phi^* R. \tag{4}$$

A solution to (1) is then given by

$$\hat{g} = (\Phi^{-1})^*(g + h).$$

In local coordinates, equation (4) is equivalent to solving

$$\begin{aligned} &\frac{1}{2}(-g^{pq}\partial^2_{pq}h_{ij} + \partial^2_{ip}h_{jq} + \partial^2_{jp}h_{iq} - \partial^2_{ij}h_{pq}) + \text{ lower order terms in } h \\ &\qquad\qquad = \partial_i\Phi^p\partial_j\Phi^q R_{pq} - \mathrm{Ricci}(g)_{ij}. \end{aligned} \tag{5}$$

DeTurck [1] showed that if $R = R_{ij}\,dx^i\,dx^j$ is a nondegenerate quadratic form, then (5) is an underdetermined elliptic system of PDE's and therefore has a local smooth solution if R is smooth and a local analytic solution if R is analytic.

In particular, he found a way to convert (5) into a determined semilinear elliptic system of PDE's. The existence of analytic solutions then follows by Cauchy-Kovalevski and smooth solutions by the standard elliptic regularity estimates and contraction mapping argument on a Banach space.

The idea is, assuming R to be nondegenerate, to set Φ equal to an appropriately chosen function of g and its derivatives, substitute this into (5) and show that the resulting equations comprise a determined elliptic system of PDE's.

DeTurck's trick is to set

$$\Phi^i(x) = x^i - \frac{1}{2}(R^{-1})^{ik}g^{pq}\left(\partial_p h_{kq} - \frac{1}{2}\partial_k h_{pq}\right). \tag{6}$$

If $|\partial^2 h|$ is small enough, then Φ is a local diffeomorphism (i.e., immersion). Also, note that the right side is the Bianchi operator (i.e., the differential operator that is applied to the Ricci curvature in the Bianchi identity). DeTurck explains why this is not a coincidence.

The act of defining Φ either using an explicit formula, as we do here, or as the solution of a system of PDE's is called *choosing or fixing a gauge*. For example, DeTurck and Kazdan showed that the optimal regularity for solution $g + h$ to (4) can be obtained using the harmonic gauge, where Φ is harmonic with respect to the metric g.

A straightforward calculation shows that

$$(\Phi^* R)_{ij} = R_{ij} + \frac{1}{2}(\partial^2_{iq}h_{pj} - \partial^2_{pj}h_{iq} - \partial^2_{ij}h_{pq}) + \text{ lower order terms in } h.$$

Substituting this into (5), we get

$$-\frac{1}{2}g^{pq}\partial^2_{pq}h_{ij} + \text{ lower order terms in } h = R_{ij} - \mathrm{Ricci}(g)$$

This is now clearly an elliptic system of PDE's for h that can be solved using Cauchy-Kovalevski if g and R are assumed to be real analytic and ellptic PDE theory if g and R are smooth. Also, if $|R - \mathrm{Ricci}(g)|$ is sufficiently small uniformly, then so is $|\partial^2 h|$ and therefore Φ, as defined by (6), is a local diffeomorphism.

There is an analogous trick for solving, given $\lambda \in \mathbb{R}$ and a nondegenerate tensor T,

$$\mathrm{Ricci}(g) + \lambda S g = T,$$

where S is the scalar curvature, *except when*

$$\lambda = -\frac{1}{2(n-1)}.$$

There is an additional degeneracy in this case, because the equation is, modulo lower order terms, equivariant with respect to conformal transformations of the metric g. By introducing one more unknown function, corresponding to the conformal factor, the system can be converted into a deteremined system of PDE's but one that is not elliptic. DeTurck and I [2] showed that it is of real principal type. By using the smooth tame estimates for PDE's of real principal type proved by Jonathan Goodman and me [3] and Nash-Moser implicit function theorem, local solutions exist.

References

[1] D. M. DeTurck. *Existence of metrics with prescribed Ricci curvature: local theory*. Invent. Math. 65.1 (1981/82), pp. 179–207.

[2] D. DeTurck and D. Yang. *Local existence of smooth metrics with prescribed curvature*. Nonlinear problems in geometry (Mobile, Ala., 1985). Vol. 51. Contemp. Math. Providence, RI: Amer. Math. Soc., 1986, pp. 37–43.

[3] J. Goodman and D. Yang. *Local solvability of nonlinear partial differential equations of real principal type*. 1988.