# MATH-UA 377 Differential Geometry: Gauss-Bonnet Theorem

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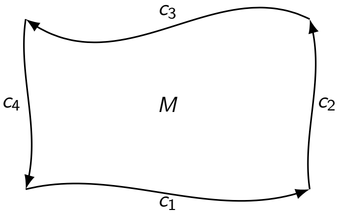
# START RECORDING
# LIVE TRANSCRIPTION

# Rectangular 2-Manifold With Boundary




- A rectangular 2-manifold $M$ with boundary consists of a set $M$ and an atlas of coordinate charts $\Phi : R \to M$, where $R = [0,1] \times [0,1]$, such that if $\Phi$ and $\Psi$ are coordinate charts, then the maps

$$\Phi^{-1} \circ \Psi, \Psi^{-1} \circ \Phi : R \to R$$

are bijective and $C^1$

- The boundary of $M$ is

$$\begin{aligned} \partial M &= \Phi((\{0,1\} \times [0,1]) \cup ([0,1] \times \{0,1\})) \\ &= c_1 \cup c_2 \cup c_3 \cup c_4 \end{aligned}$$

# Orientations of $M$ and $\partial M$

- An orientation of $M$ is an orientation on $T_p M$ for each $p \in M$
- A nowhere zero 2-form $\Theta$ uniquely determines an orientation on $M$
  - A basis $(b_1, b_2)$ of $T_p M$ has positive orientation if

$$\langle b_1 \otimes b_2, \Theta(p) \rangle$$

- If $(u^1, u^2) \in R$ denote coordinates on $M$, then $du^1 \wedge du^2$ determines an orientation, where $(\partial_1 \Phi, \partial_2 \Phi)$ has positive orientation
- Any coordinate chart $\Psi : [0,1] \times [0,1] \to M$ preserves orientation if the Jacobian of $\Phi^{-1} \circ \Psi$ has positive determinant
- Orientation on $\partial M$ is compatible with the one on $M$ if, given a vector $v_1$ tangent to $\partial M$ and a vector $v_2$ that points into $M$, $(v_1, v_2)$ is a positively oriented frame for $M$

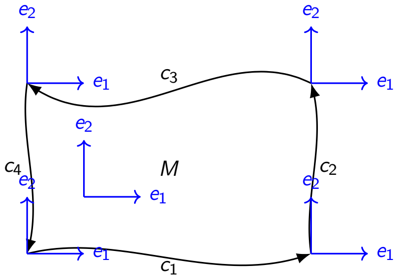
# Stokes' Theorem

## Theorem

*If $\theta$ is a $C^1$ 1-form on an oriented rectangular 2-manifold with boundary, then*

$$\int_M d\theta = \int_{\partial M} \theta$$

# Oriented Orthonormal Frame on Riemanian Rectangular 2-Manifold




- $M$ be an oriented rectangular 2–manifold with a Riemannian metric $g$
- Let $(e_1, e_2)$ be a positively oriented orthonormal frame on $M$
- Let $(\omega^1, \omega^2)$ be the dual frame
- Let $\omega_2^1 = -\omega_1^2$ be the connection 1-form

# Change of Frame on Local 2-Manifold

- Let $(e_1, e_2)$ be an oriented orthonormal frame with dual frame $(\omega^1, \omega^2)$
- Let $(f_1, f_2)$ be the frame rotated counterclockwise by angle $\alpha$ relative to $(e_1, e_2)$
- Therefore,

$$
\begin{aligned}
f_1 &= e_1 \cos\alpha + e_2 \sin\alpha \\
f_2 &= -e_1 \sin\alpha + e_2 \cos\alpha,
\end{aligned}
$$

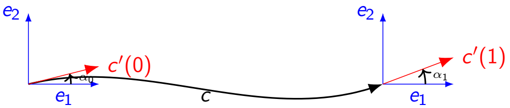
- If $(\omega^1, \omega^2)$ is the dual frame of $(e_1, e_2)$ and $(\eta^1, \eta^2)$ is the dual frame of $(f_1, f_2)$, then

$$
\begin{aligned}
\omega^1 &= (\cos\alpha)\eta^1 - (\sin\alpha)\eta^2 \\
\omega^2 &= (\sin\alpha)\eta^1 + (\cos\alpha)\eta^2
\end{aligned}
$$

- The connection 1-forms and the angle $\alpha$ satisfy

$$
\omega_2^1 = \eta_2^1 + d\alpha
$$

# Change of Angle Along Curve



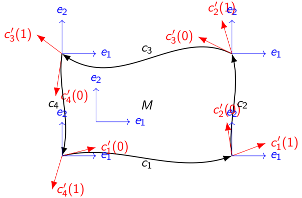

- Let

$$\begin{aligned}\alpha_0 &= \text{ counterclockwise angle from } e_1(c(0)) \text{ to } c'(0)\\ \alpha_1 &= \text{ counterclockwise angle from } e_1(c(1)) \text{ to } c'(1)\end{aligned}$$

- By the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_c \omega_2^1 &= \int_c \eta_2^1 + d\alpha\\ &= \int_{t=0}^{t=1} \langle \eta_2^1 + d\alpha, c'(t)\rangle\, dt\\ &= \int_{t=0}^{t=1} \langle \eta_2^1, c'(t)\, dt + \int_{t=0}^{t=1} \alpha'(t)\, dt\\ &= -\int_{t=0}^{t=1} \kappa_g \sigma\, dt + \alpha_1 - \alpha_0\end{aligned}$$

# Change in Angles at Vertices



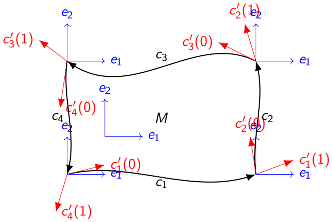

- Angles: For each $1 \leq k \leq 4$,

$$\begin{aligned} \alpha_{k,0} &= \text{ angle from } e_1(c_k(0)) \text{ to } c_k'(0) \\ \alpha_{k,1} &= \text{ angle from } e_1(c_k(1)) \text{ to } c_k'(1) \\ \theta_k &= \text{ angle from } c_k'(1) \text{ to } c_{k+1}'(0), \text{ where } c_5 = c_1 \end{aligned}$$

- Therefore,

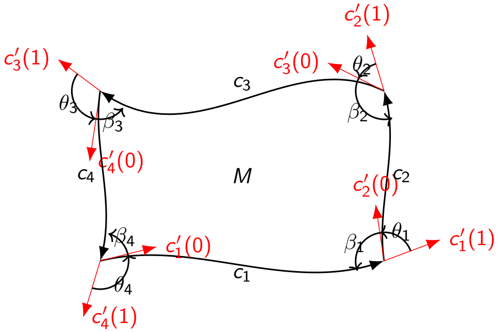
$$\begin{aligned} \theta_1 &= \alpha_{2,0} - \alpha_{1,1} \\ \theta_2 &= \alpha_{3,0} - \alpha_{2,1} \\ \theta_3 &= \alpha_{4,0} - \alpha_{3,1} \\ \theta_4 &= (2\pi + \alpha_{1,0}) - \alpha_{4,0} \end{aligned}$$

# Integral of Connection 1-Form Around Boundary of $M$




$$
\begin{aligned}
&\int_{\partial M} \omega_2^1 + \kappa_g \, ds \\
&= \alpha_{11} - \alpha_{10} + \alpha_{21} - \alpha_{20} + \alpha_{31} - \alpha_{30} + \alpha_{41} - \alpha_{40} \\
&= -((\alpha_{20} - \alpha_{11}) + (\alpha_{30} - \alpha_{21}) + (\alpha_{40} - \alpha_{31}) + (\alpha_{10} - \alpha_{41})) \\
&= -(\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi) \\
&= 2\pi - (\theta_1 + \theta_2 + \theta_3 + \theta_4)
\end{aligned}
$$

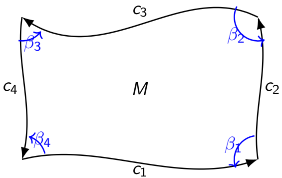
# Integral of Connection 1-Form Around Boundary




- Interior angle: $\beta_k = \pi - \theta_k$

$$\int_c (\omega_2^1 + \kappa_g \, dt) = 2\pi - (\theta_1 + \theta_2 + \theta_3 + \theta_4)$$
$$= 2\pi - (4\pi - \beta_1 - \beta_2 - \beta_3 - \beta_4)$$
$$= -2\pi + \beta_1 + \beta_2 + \beta_3 + \beta_4$$

# Gauss-Bonnet Theorem for Riemannian Rectangular 2-Manifold

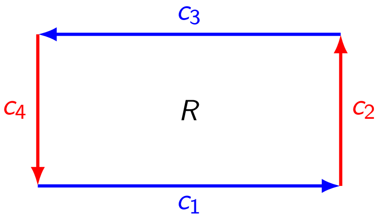



- Given an oriented Riemannian rectangular 2-manifold $M$ with boundary,

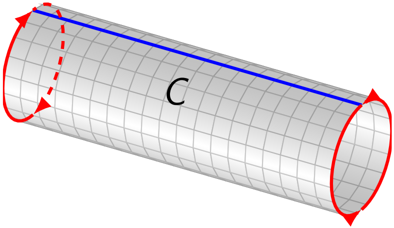
$$\int_S K\,dA + \int_c \kappa_g\,dt = -2\pi + \beta_1 + \beta_2 + \beta_3 + \beta_4,$$

  where $K$ is the Gauss curvature, $\kappa_g$ is the geodesic curvature, and $\beta_1, \beta_2, \beta_3, \beta_4$ are the interior angles at the vertices
- Although frame and connection form are needed for proof, they do not appear in the theorem
- Everything in formula is defined globally on $M$

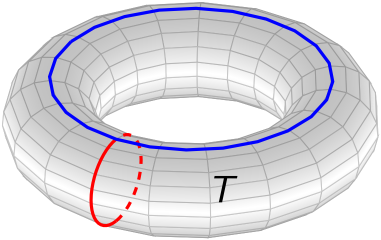
# Shapes by Gluing Rectangles




$$\partial R = c_1 \cup c_2 \cup c_3 \cup c_4$$

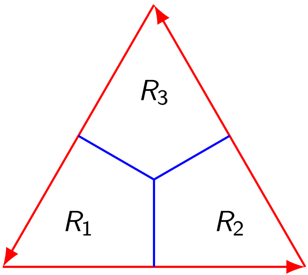



$$\partial C = c_2 \cup c_4$$



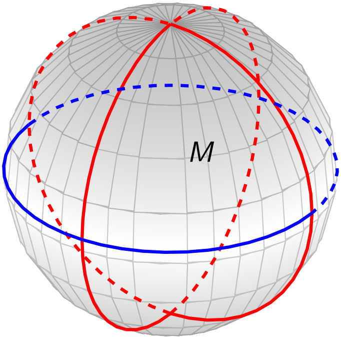

$$\partial T = \emptyset$$

# Triangle by Gluing Rectangles




$$T = R_1 \cup R_2 \cup R_3$$

# Triangulation of Sphere $\implies$ Rectangulation of Sphere




$$S = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_6 \cup T_7 \cup T_8$$
$$\partial S = \emptyset$$

# Geodesic Curvature Independent of Orientation

- Let $c : [a, b] \to M$ be an oriented curve and $\tilde{c} : [b, a] \to M$ be the same curve with the opposite orientation
- If $(f_1, f_2)$ is an adapted oriented orthonormal frame along $c$, then the geodesic curvature of $c$ is

$$\kappa_g = f_2 \cdot f_1'$$

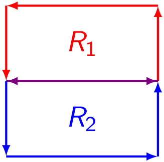
- The frame $(\tilde{f}_1, \tilde{f}_2) = (-f_1, -f_2)$ has the same orientation in $M$ as $(f_1, f_2)$ and therefore is an adapted oriented frame along $\tilde{c}$
- The geodesic curvature of $\tilde{c}$ is therefore

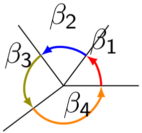
$$\tilde{\kappa}_g = \tilde{f}_2 \cdot \tilde{f}_1' = (-f_2) \cdot (-f_1)' = f_2 \cdot f_1' = \kappa_g$$

- Therefore,

$$\int_{\tilde{c}} \tilde{\kappa}_g = -\int_c \kappa_g$$

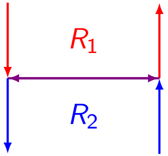
# Closed Riemannian 2-Manifold



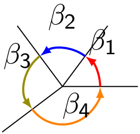




- Edge of a rectangular 2-manifold can be glued to an edge of another rectangular 2-manifold if
  - The edges have the same length
  - The geodesic curvature, as a function of arclength, is the same on both edges
- A closed 2-manifold is a finite collection of rectangular 2-manifolds satisfying the following:
  - Each edge of a rectangular manifold is glued to another edge of a rectangular manifold
  - At each vertex, the sum of the interior angles is $2\pi$
- An orientation on a closed 2-manifold is an orientation on each rectangle such that each edge has two opposite orientations

# Towards Gauss-Bonnet for a Closed Surface




Each edge is in boundary of two rectangles




Sum of angles at each vertex $= 2\pi$

- Let $M$ be a closed surface and

$$\begin{aligned} F &= \text{ number of faces (rectangles)} \\ E &= \text{ number of edges} \\ V &= \text{ number of vertices} \end{aligned}$$

- Since each face has 4 edges, and each edge is counted twice,

$$2E = 4F \implies F = F - E$$

- If $\beta_{1,1}, \ldots, \beta_{N,4}$ are all interior angles, then

$$\beta_1 + \cdots + \beta_N = 2\pi V$$

# Gauss-Bonnet on a Closed Surface

- Consider a rectangulated closed 2-manifold $M = R_1 \cup \cdots \cup R_F$
- Gauss-Bonnet for rectangular 2-manifolds $\implies$

$$\begin{aligned}
\int_M K \, dA &= \sum_{k=1}^{N} \int_{R_k} K \, dA \\
&= \sum_{k=1}^{F} \left( \int_{\partial R_k} \kappa_g \, dt + \beta_{k,1} + \beta_{k,2} + \beta_{k,3} + \beta_{k,4} - 2\pi \right) \\
&= 2\pi V - 2\pi F \\
&= 2\pi (V - E + F)
\end{aligned}$$

# Consequences of Gauss-Bonnet

- Given any two Riemannian metrics on a closed 2-manifold $M$,

$$\int_M K_1 \, dA_1 = \int_M K_2 \, dA_2$$

- Given any two rectangulations of a closed 2-manifold $M$,

$$V_1 - E_1 + F_1 = V_2 - E_2 + F_2$$

- Define the Euler characteristic of $M$ to be

$$\chi(M) = V - E + F = \frac{1}{2\pi} \int_M K \, dA$$

- It depends on neither the Riemannian metric nor the rectangulation
- It therefore is a topological invariant

# Gauss-Bonnet Theorem for a Polygonal Surface

- Let $M$ be a surface parameterized by a polygon $P$ with $N$ edges
- Let $\theta_1, \ldots, \theta_N$ be the exterior angles at the vertices
- The interior angles are therefore $\beta_j = \pi - \theta_j$, $1 \leq j \leq N$
- The Gauss-Bonnet theorem says:

$$\begin{aligned}\int_M K\,dA &= -\int_{\partial P} \kappa_g\,dt + 2\pi - (\theta_1 + \cdots + \theta_N) \\ &= -\int_{\partial P} \kappa_g\,dt + 2\pi - (\pi - \beta_1 + \cdots + \pi - \beta_N) \\ &= -\int_{\partial P} \kappa_g\,dt + (2 - N)\pi + \beta_1 + \cdots + \beta_N\end{aligned}$$

# Decomposition of Surface into Polygonal Surfaces

- $M = P_1 \cup \cdots \cup P_F$, where
  - $P_k$ is parameterized by a polygon with $E_k$ sides
  - $\beta_{k,1}, \ldots, \beta_{k,E_k}$ are interior angles
  - $P_j \cap P_k$ is a union of edges and vertices of $P_j$ and $P_k$
- The number of edges and faces must satisfy

$$2E = \sum_{k=1}^{F} E_k$$

- The sum of all interior angles of all vertices must satisfy

$$\sum_{k=1}^{F} \sum_{j=1}^{E_k} \beta_{k,j} = 2\pi V$$

- The sum of the integrals of geodesic curvature along the edges must add to zero

# Gauss-Bonnet Theorem

- Gauss-Bonnet for a polygonal surface with $N$ edges

$$\int_P K\,dA = -\int_{\partial P} \kappa_g\,dt + (2-N)\pi + \sum_{j=1}^{N} \beta_j$$

- Therefore,

$$\begin{aligned}
\int_M K\,dA &= \sum_{k=1}^{F} \int_{P_k} K\,dA \\
&= \sum_{k=1}^{F} \left( -\int_{\partial P_k} \kappa_g\,dt + (2-E_k)\pi + \sum_{j=1}^{E_k} \beta_{k,j} \right) \\
&= 2\pi(F - E + V)
\end{aligned}$$