MATH-UA 377 Differential Geometry: Gauss-Bonnet Theorem for Riemannian Rectangular 2-Manifold

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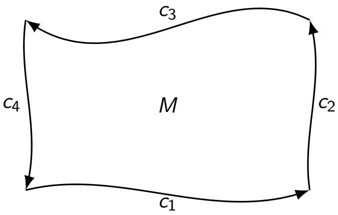
LIVE TRANSCRIPTION

Rectangular 2-Manifold With Boundary



- A rectangular 2-manifold M with boundary consists of the following
  - A set M
  - A bijective map $\Phi : [0,1] \times [0,1] \to M$
- A $\Psi : [0,1] \times [0,1] \to M$ is a coordinate chart if the maps

$$\Phi^{-1} \circ \Psi, \Psi^{-1} \circ \Phi : [0,1] \times [0,1] \to [0,1] \times [0,1]$$

  are bijective and C^1
- The boundary of M is

$$\begin{aligned} \partial M &= \Phi((\{0,1\} \times [0,1]) \cup ([0,1] \times \{0,1\})) \\ &= c_1 \cup c_2 \cup c_3 \cup c_4 \end{aligned}$$

Orientation of Manifold and its Boundary

- An orientation of M is given by the frame $(\partial_1\Phi, \partial_2\Phi)$ of T_*M
- Equivalently, the orientation is given by $du^1 \wedge du^2$, where (u^1, u^2) are the coordinates on M
- Any coordinate chart $\Psi : [0,1] \times [0,1] \to M$ preserves orientation if the Jacobian of $\Phi^{-1} \circ \Psi$ has positive determinant
- Orientation on ∂M is compatible with the one on M if, given a vector v_1 tangent to ∂M and a vector v_2 that points into M, (v_1, v_2) is a positively oriented frame for M

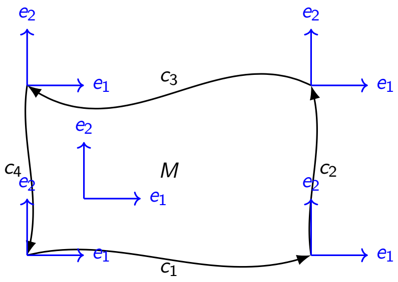
Stokes' Theorem

Theorem

If θ *is a* C^1 1*-form on an oriented rectangular* 2*-manifold with boundary, then*

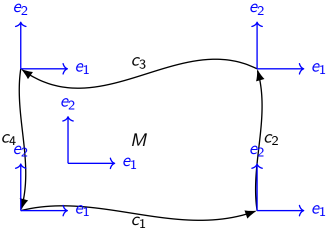
$$\int_M d\theta = \int_{\partial M} \theta$$

Oriented Orthonormal Frame on Riemanian Rectangular 2-Manifold



- M be an oriented rectangular 2–manifold with a Riemannian metric g
- Let (e_1, e_2) be a positively oriented orthonormal frame on M
- Let (ω^1, ω^2) be the dual frame
- Let $\omega_2^1 = -\omega_1^2$ be the connection 1-form

Integral of Gauss Curvature



- By Stokes' Theorem,

$$\begin{aligned}\int_M K\,\omega^1 \wedge \omega^2 &= \int_M d\omega_2^1 \\ &= \int_{\partial M} \omega_2^1 \\ &= \int_{c_1} \omega_2^1 + \int_{c_2} \omega_2^1 + \int_{c_3} \omega_2^1 + \int_{c_4} \omega_2^1\end{aligned}$$

- Need geometric interpretation of the line integrals on the right

Geodesic and Normal Curvature of a Curve in Surface

- Consider
  - M is a surface $S \subset \mathbb{E}^3$
  - g is the first fundamental form,
  - $c : I \to M$ is a C^1 curve
- On one hand, if $\sigma = |c'| = \sqrt{g(c', c')}$ is the speed,

$$f_1' = \sigma(f_2\kappa_g + f_3\kappa_n)$$

- On the other hand,

$$\begin{aligned} f_1' &= \langle c', df_1 \rangle \\ &= \langle c', f_2\,\eta_1^2 + f_3\,\eta_1^3 \rangle \\ &= f_2\langle c',\, \eta_1^2 \rangle + f_3\langle c',\, \eta_1^3 \rangle \end{aligned}$$

- Therefore,

$$\sigma\kappa_g = \langle c', \eta_1^2 \rangle$$

Geodesic Curvature of a Curve in a Riemannian 2-Manifold

- Let g be a Riemannian metric on a rectangular 2-manifold M with boundary
- Consider a curve $c : [0, 1] \to M$
- Let (f_1, f_2) be a positively oriened orthonormal frame along c, where
$$f_1 = \frac{c'}{\sigma}$$
- Let (η^1, η^2) be the dual frame and η^2_1 the connection 1-form
- The geodesic curvature of the curve c is defined to be

$$\kappa_g = \langle \eta^2_1, f_1 \rangle = \frac{\langle \eta^2_1, c' \rangle}{\sigma}$$

Change of Frame on Local 2-Manifold

- Let (e_1, e_2) be an oriented orthonormal frame
- Let (f_1, f_2) be the frame rotated counterclockwise by angle α relative to (e_1, e_2)
- Therefore,

$$
\begin{aligned}
f_1 &= e_1 \cos\alpha + e_2 \sin\alpha \\
f_2 &= -e_1 \sin\alpha + e_2 \cos\alpha,
\end{aligned}
$$

- IF (ω^1, ω^2) is the dual frame of (e_1, e_2) and (η^1, η^2) is the dual frame of (f_1, f_2), then

$$
\begin{aligned}
\omega^1 &= (\cos\alpha)\eta^1 - (\sin\alpha)\eta^2 \\
\omega^2 &= (\sin\alpha)\eta^1 + (\cos\alpha)\eta^2
\end{aligned}
$$

Change of Connection 1-Form

Differentiating

$$\omega^1 = (\cos\alpha)\eta^1 - (\sin\alpha)\eta^2$$
$$\omega^2 = (\sin\alpha)\eta^1 + (\cos\alpha)\eta^2$$

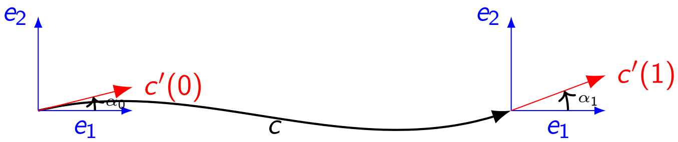
we get

$$\begin{aligned}
d\omega^1 &= (\cos\alpha)\, d\eta^1 - (\sin\alpha)\, d\alpha\wedge\eta^1 - (\sin\alpha)\, d\eta^2 - (\cos\alpha)\, d\alpha\wedge\eta^2 \\
&= -\eta^1_2\wedge(\eta^2\cos\alpha) + \eta^2_1\wedge(\eta^1\sin\alpha) + d\alpha\wedge(-\eta^1\sin\alpha - \eta^2\cos\alpha) \\
&= -\eta^1_2\wedge(\eta^2\cos\alpha + \eta^1\sin\alpha) - d\alpha\wedge\omega^2 \\
&= -(\eta^1_2 + d\alpha)\wedge\omega^2 \\
d\omega^2 &= (\sin\alpha)\, d\eta^1 + (\cos\alpha)\, d\alpha\wedge\eta^1 + (\cos\alpha)d\eta^2 - (\sin\alpha)\, d\alpha\wedge\eta^2 \\
&= \eta^2_1\wedge(-\eta^1\cos\alpha + \eta^2\sin\alpha) + d\alpha\wedge(\eta^1\cos\alpha - \eta^2\sin\alpha) \\
&= -(\eta^2_1 - d\alpha)\wedge\omega^2
\end{aligned}$$

Therefore,

$$\eta^1_2 = \omega^1_2 - d\alpha \text{ and } \eta^2_1 = \omega^2_1 + d\alpha$$

Change of Angle Along Curve



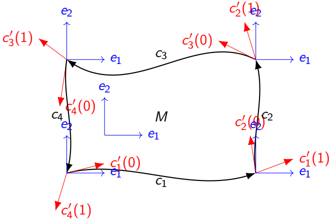
- Let

$$\alpha_0 = \text{ counterclockwise angle from } e_1(c(0)) \text{ to } c'(0)$$
$$\alpha_1 = \text{ counterclockwise angle from } e_1(c(1)) \text{ to } c'(1)$$

- By the Fundamental Theorem of Calculus,

$$\begin{aligned} \alpha_1 - \alpha_0 &= \int_{t=0}^{t=1} \alpha'(t)\,dt = \int_{t=0}^{t=1} \langle c'(t), d\alpha \rangle\,dt \\ &= \int_{t=0}^{t=1} \langle c'(t), \omega_2^1 \rangle - \langle c'(t), \eta_2^1 \rangle\,dt \\ &= \int_c \omega_2^1 + \int_{t=0}^{t=1} \sigma \kappa_g\,dt \end{aligned}$$

Change in Angles at Vertices



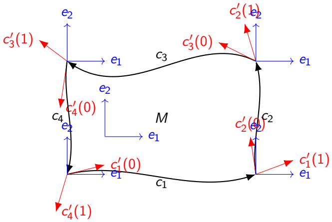
- Angles: For each $1 \leq k \leq 4$,

$$\begin{aligned} \alpha_{k,0} &= \text{ angle from } e_1(c_k(0)) \text{ to } c_k'(0) \\ \alpha_{k,1} &= \text{ angle from } e_1(c_k(1)) \text{ to } c_k'(1) \\ \theta_k &= \text{ angle from } c_k'(1) \text{ to } c_{k+1}'(0), \text{ where } c_5 = c_1 \end{aligned}$$

- Therefore,

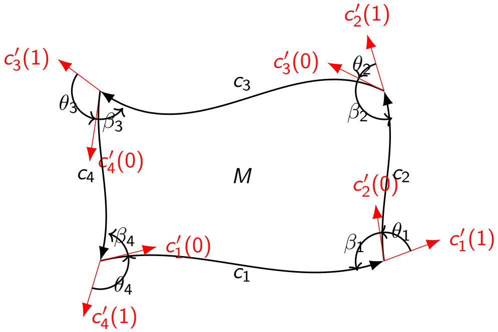
$$\begin{aligned} \theta_1 &= \alpha_{2,0} - \alpha_{1,1} \\ \theta_2 &= \alpha_{3,0} - \alpha_{2,1} \\ \theta_3 &= \alpha_{4,0} - \alpha_{3,1} \\ \theta_4 &= (2\pi + \alpha_{1,0}) - \alpha_{4,0} \end{aligned}$$

Integral of Connection 1-Form Around Boundary of M



$$
\begin{aligned}
&\int_{\partial M} (\sigma \kappa_g \, dt + \omega_2^1) \\
&= \alpha_{11} - \alpha_{10} + \alpha_{21} - \alpha_{20} + \alpha_{31} - \alpha_{30} + \alpha_{41} - \alpha_{40} \\
&= -((\alpha_{20} - \alpha_{11}) + (\alpha_{30} - \alpha_{21}) + (\alpha_{40} - \alpha_{31}) + (\alpha_{10} - \alpha_{41})) \\
&= -(\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi) \\
&= 2\pi - (\theta_1 + \theta_2 + \theta_3 + \theta_4)
\end{aligned}
$$

Integral of Connection 1-Form Around Boundary



- Interior angle: $\beta_k = \pi - \theta_k$

$$\begin{aligned}\int_c (\omega_2^1 + \kappa_g \, dt) &= 2\pi - (\theta_1 + \theta_2 + \theta_3 + \theta_4) \\ &= 2\pi - (4\pi - \beta_1 - \beta_2 - \beta_3 - \beta_4) \\ &= -2\pi + \beta_1 + \beta_2 + \beta_3 + \beta_4\end{aligned}$$

Gauss-Bonnet Theorem for Riemannian Rectangular 2-Manifold

- $$\begin{aligned}\int_M K\omega^1 \wedge \omega^2 = \int_M d\omega_2^1 = \int_{\partial M} M\omega_2^1 \\ = -\int_{\partial M} \sigma\kappa_g \, dt - 2\pi + \beta_1 + \beta_2 + \beta_3 + \beta_4\end{aligned}$$
- If $dA = \omega^1 \wedge \omega^2$,
  $$\int_S K \, dA = -\int_c \kappa_g \, dt - 2\pi + \beta_1 + \beta_2 + \beta_3 + \beta_4$$
- Although frame and connection form are needed for proof, they do not appear in the theorem
- Everything in formula is defined globally on M

Gauss-Bonnet Theorem for an Oriented Rectangular 2-Manifold with Boundary

- Given
  - Oriented Riemannian rectangular 2-manifold M with boundary
  - $dA = \omega^1 \wedge \omega^2$
  - κ_g = geodesic curvature of boundary
  - $\beta_1, \beta_2, \beta_3, \beta_4$ = interior angles at vertices,

  the following holds:

$$\int_M K\, dA + \int_{\partial M} \kappa_g\, dt = -2\pi + \beta_1 + \beta_2 + \beta_3 + \beta_4$$

- Similar theorem holds for any Riemannian polygonal 2-manifold with boundary