MATH-UA 377 Differential Geometry: Riemannian metric Orthonormal Frame Uniqueness of Connection 1-Form Adapted Orthonormal Frame for Curve in Surface Geodesic Curvature

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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START RECORDING
LIVE TRANSCRIPTION

Local 2-Manifold and Associated Bundles

- A local C^k 2-manifold consists of the following:
  - A set S
  - An atlas of C^k coordinate maps $\Phi : D \to S$
- Tangent bundle $T_* S$
  - $T_p S$ is the vector space of velocity vectors of curves passing through p
  - A vector field V assigns to each $p \in S$ a tangent vector $V(p) \in T_p S$
- Cotangent bundle $T^* S$
  - $T_p^* S = (T_p S)^*$ is the vector space of 1-tensors of $T_p S$
  - A 1-form θ assigns to each $p \in S$ a 1-tensor $\theta(p) \in T_p^* S$
- Exterior 2-tensor bundle $\Lambda^2 T^* S$
  - $\Lambda^2 T_p^* S$ is the vector space of exterior 2-tensors on $T_p S$
  - A 2-form Θ assigns to each $p \in S$ an exterior 2-tensor $\Theta(p) \in \Lambda^2 T_p^* S$
- Symmetric 2-tensor bundle $S^2 T^* S$
  - $S^2 T_p^* S$ is the vector space of symmetric 2-tensors of $T_p S$
  - A symmetric 2-tensor field h assigns to each $p \in S$ a symmetric 2-tensor $h(p) \in S^2 T_p^* S$

Orientation of a Local 2-Manifold

- An orientation on a local 2-manifold is an orientation on T_pS that depends continuously on p
- An orientation can be specified using a frame (v_1, v_2) of vector fields
  - $(v_1(p), v_2(p))$ has positive orientation on T_pS
- An orientation can also be specified by a nowhere vanishing 2-form Θ on S
  - A basis (v_1, v_2) of T_pS has positive orientation if

$$\langle v_1 \otimes v_2, \Theta(p) \rangle > 0$$

- If (e_1, e_2) is a positively oriented frame, and (ω^1, ω^2) is the dual frame, then $\omega^1 \wedge \omega^2$ is also positively oriented

Riemannian Metric on Local 2-Manifold

- A Riemannian metric g on a local 2-manifold S is a positive definite symmetric 2-tensor field
- At each point $p \in S$, $g(p)$ defines a dot product on T_pS
  - A tangent vector $v \in T_pS$ has a length $|v|$, where $|v|^2 = g(p)(v, v)$
  - Two tangent vectors $v, w \in T_pS$ have an angle θ between them, where
$$g(p)(v, w) = |v||w| \cos\theta$$
- The length of a curve $c : I \to S$ is
$$\ell = \int_I |c'(t)|\, dt$$
- Example: First fundamental form of a local surface $S \subset \mathbb{E}^3$
  - $g(p)$ is the dot product on $\mathbb{V}^3$ restricted to T_pS

Riemannian Metric With Respect to Coordinates

- Let $\Phi : D \to S$ be a coordinate map
- Denote the coordinates on D by (u^1, u^2)
- Let (∂_1, ∂_2) be the standard basis of $\widehat{\mathbb{R}}^2$
- Let (du^1, du^2) be the dual basis
- Given $p = \Phi(u^1, u^2)$, $v = v^1\partial_1 + v^2\partial_2$, $w = w^1\partial_1 + w^2\partial_2$,

$$\begin{aligned}
g(p)(v, w) &= g(p)(v^1\partial_1 + v^2\partial_2, w^1\partial_1 + w^2\partial_2) \\
&= v^1 w^1 g(p)(\partial_1, \partial_1) + v^1 w^2 g(p)(\partial_1, \partial_2) \\
&\quad + v^2 w^1 g(p)(\partial_2, \partial_1) + v^2 w^2 g(p)(\partial_2, \partial_2) \\
&= g_{11}\langle v \otimes w, du \otimes du\rangle + g_{12}\langle v \otimes w, du^1 \otimes du^2\rangle \\
&\quad + g_{21}\langle v \otimes w, du^2 \otimes du^1 + g_{22}\langle v \otimes w, du^2 \otimes du^2\rangle \\
&= \langle v \otimes w, g_{jk}\, du^j \otimes du^k\rangle
\end{aligned}$$

where

$$g_{jk} = g(\partial_j, \partial_k)$$

Orthonormal Frame of an Oriented Local 2-Manifold

- Let S be an oriented local 2-manifold with a Riemannian metric g
- A frame (e_1, e_2) is an adapted orthonormal frame, if
  - $(e_1(p), e_2(p))$ has positive orientation on T_pS
  - $(e_1(p), e_2(p))$ is an orthonormal basis for the dot product $g(p)$

$$g(p)(e_1(p), e_1(p)) = g(p)(e_2(p), e_2(p)) = 1$$
$$g(p)(e_1(p), e_2(p)) = g(p)(e_2(p), e_1(p)) = 0$$

- If (ω^1, ω^2) is the dual frame, then
  - $\omega^1 \wedge \omega^2$ has positive orientation
  - The metric can be written as

$$g = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$$

because

$$\begin{aligned} g(v^1 e_1 + v^2 e_2, w^1 e_1 + w^2 e_2) &= v^1 w^1 + v^2 w^2 \\ &= \langle v \otimes w, \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 \rangle \end{aligned}$$

Connection 1-Form

- Given an orthonormal coframe (ω^1, ω^2), there is a unique 1-form $\omega_2^1 = -\omega_1^2$ such that

$$\begin{aligned} d\omega^1 + \omega_2^1 \wedge \omega^2 &= 0 \\ d\omega^2 + \omega_1^2 \wedge \omega^1 &= 0 \end{aligned} \tag{1}$$

- Proof
  - Suppose

$$\begin{aligned} d\omega^1 &= a\,\omega^1 \wedge \omega^2 \\ d\omega^2 &= b\,\omega^2 \wedge d\omega^1 \end{aligned}$$

  - If

$$\omega_2^1 = p\,\omega^1 + q\,\omega^2,$$

    then

$$\begin{aligned} d\omega^1 + \omega_2^1 \wedge \omega^2 &= (a+p)\,\omega^1 \wedge \omega^2 \\ d\omega^2 + \omega_1^2 \wedge \omega^1 &= (b-q)\,\omega^2 \wedge \omega^1 \end{aligned}$$

  - It follows that equations (1) hold if and only if $p = -a$ and $q = b$

Gauss Curvature

- $d\omega^1_2$ is a 2-form on S and therefore a scalar multiple of $\omega^1 \wedge \omega^2$
- The Gauss curvature of S is the scalar function K such that

$$d\omega^1_2 = K\omega^1 \wedge \omega^2$$

Back to Surface in $\mathbb{E}^3$

- Let g be the first fundamental form
- Let (e_1, e_2, e_3) be the adapted frame and $(\omega^1, \omega^2, \omega^2)$ the dual frame
- Recall that there are 1-forms $\omega^j_k = -\omega^k_j$ such that

$$
\begin{aligned}
de_1 &= e_2\omega^2_1 + e_3\omega^3_1 \\
de_2 &= e_1\omega^1_2 + e_3\omega^3_2 \\
de_3 &= e_1\omega^1_3 + e_2\omega^2_3
\end{aligned}
$$

- These forms satisfy the structure equations, which include

$$
\begin{aligned}
d\omega^1 + \omega^1_2 \wedge \omega^2 &= 0 \\
d\omega^2 + \omega^2_1 \wedge \omega^1 &= 0 \\
d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 &= 0
\end{aligned}
$$

- Compare this to the orthonormal frame, dual frame, structure equations for a Riemannian metric on a local 2-manifold

Surface in $\mathbb{E}^3$ as a Local Riemannian 2-Manifold

- (e_1, e_2), (ω^1, ω^2) are the same
- ω_2^1 is the same connection 1-form
- Recall that the second fundamental form can be written as a symmetric matrix II, where

$$\omega_1^3 = \mathrm{II}_{11}\omega^1 + \mathrm{II}_{12}\omega^2$$
$$\omega_2^3 = \mathrm{II}_{21}\omega^1 + \mathrm{II}_{22}\omega^2$$

- Since $d\omega_2^1 + \omega_3^1 \wedge \omega_2^3 = 0$,

$$\begin{aligned} d\omega_2^1 &= \omega_1^3 \wedge \omega_2^3 \\ &= (\mathrm{II}_{11}\omega^1 + \mathrm{II}_{12}\omega^2) \wedge (\mathrm{II}_{21}\omega^1 + \mathrm{II}_{22}\omega^2) \\ &= (\mathrm{II}_{11}\mathrm{II}_{22} - \mathrm{II}_{12}\mathrm{II}_{21})\omega^1 \wedge \omega^2 \\ &= (\det \mathrm{II})\omega^1 \wedge \omega^2 \end{aligned}$$

- Since $d\omega_2^1 = K\omega^1 \wedge \omega^2$, the Gauss curvature K for a surface in $\mathbb{E}^3$ satisfies

$$K = \det \mathrm{II}$$

Curve in an Oriented Surface in $\mathbb{E}^3$

- Consider a unit speed curve $c : I \to S \subset \mathbb{E}^3$
- There is a unique orthonormal frame (f_1, f_2, f_3) along the curve such that
  - $f_1 = c'$
  - (f_1, f_2) is a positively oriented frame on S
  - f_3 is a positively oriented unit normal for S
- Since $f_1 \cdot c'' = f_1 \cdot f_1' = 0$, there are scalar functions κ_g and κ_n such that

$$c'' = \kappa_g f_2 + \kappa_n f_3,$$

where

$$\kappa = |f_1'| = \sqrt{\kappa_g^2 + \kappa_n^2}$$

is the curvature of c as a curve in $\mathbb{E}^3$

Structure Equations for Curve in Oriented Surface

- If (η^1, η^2, η^3) is the dual frame and η^j_k the connection 1-forms, then

$$\begin{aligned} c'' &= f_1' \\ &= \langle c', df_1 \rangle \\ &= \langle c', f_2\eta^2_1 + f_3\eta^3_1 \rangle \\ &= f_2\langle c', \eta^2_1 \rangle + f_3\langle c', \eta^3_1 \rangle \end{aligned}$$

- The normal curvature $\kappa_n = \langle c', \eta^3_1 \rangle$
  - Measures how fast c' is turning toward f_3
  - Depends only on the second fundamental form in the direction c'
- The geodesic curvature $\kappa_g = \langle c', \eta^2_1 \rangle$
  - Measures how fast c' is turning toward f_2
  - Depends only on the Riemannian metric and its connection 1-form

Geodesic Curvature of a Curve in an Oriented Riemannian Local 2-Manifold

- Given a unit speed curve c in an oriented Riemannian Local 2-Manifold S, there is a unique positively oriented orthonormal frame (f_1, f_2) such that $f_1 = c'$
- Let (η^1, η^2) be the dual frame and $\eta^1_2 = -\eta^2_1$ the connection 1-form
- Define the geodesic curvature of c to be the function

$$\kappa_g(t) = \langle c'(t), \eta^2_1 \rangle$$