MATH-UA 377 Differential Geometry: Local 2-Manifold Atlas of Coordinate Charts Tangent and Cotangent Bundles Riemannian metric Adapted Orthonormal Frame

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START RECORDING
LIVE TRANSCRIPTION

Local 2-Manifold

- A local 2-manifold consists of the following:
  - A set S
  - A bijective map

    $$\Phi : D \to S,$$

    where D is an open subset of $\mathbb{R}^2$
- Another bijective map $\Psi : D' \to S$, where $D' \subset \mathbb{R}^2$ is open, is a compatible C^k coordinate map, if the maps

  $$\Psi^{-1} \circ \Phi : D \to D'$$
  $$\Phi^{-1} \circ \Psi : D' \to D$$

  are both C^k maps
- The set of all compatible C^k coordinate maps is called the maximal atlas of S
- Combination of S with a C^k maximal atlas is called a C^k local 2-manifold

Curves

- A map $c : I \to S$, where I is a connected open interval, is a C^k curve if, for any coordinate map $\Phi : D \to S$, the map

$$\Phi^{-1} \circ c : I \to D \subset \mathbb{R}^2$$

  is a C^k map
- If this holds for one C^k coordinate map, it holds for any other one
  - Follows by the chain rule

Tangent Space is Space of Velocity Vectors

- The tangent space at $p \in S$ is defined to be the space of all possible velocity vectors of curves
- Given $p \in S$ and a coordinate map $\Phi : D \to S$ such that $\Phi(0) = p$, there is a map

$$\Phi_* : \widehat{\mathbb{R}}^2 \to T_p S,$$

- If $\hat{v} \in \widehat{\mathbb{R}}^2$, let $\hat{c} : I \to D$ be a curve such that $\hat{c}(0) = 0$ and $\hat{c}'(0) = \hat{v}$ and define

$$\Phi_* \hat{v} = (\Phi \circ \hat{c})'(0)$$

- Conversely, given $v \in T_p S$, there is a C^k curve $c : I \to S$ such that $c(0) = p$ and $c'(0) = v$
- If $\hat{v} = (\Phi^{-1} \circ c)'(0)$, then $\Phi_* \hat{v} = v$
- Let $\Phi_*^{-1} = (\Phi_*)^{-1} : T_p S \to \widehat{\mathbb{R}}^2$

Tangent Space is a Vector Space

- If $\Psi : D' \to S$ is another coordinate map, then
$$\begin{aligned} \Psi_*^{-1} \circ \Phi_* : \widehat{\mathbb{R}}^2 &\to \widehat{\mathbb{R}}^2 \\ \hat{v} = \hat{c}'(0) &\mapsto (\Psi^{-1} \circ \Phi \circ \hat{c})'(0) \end{aligned}$$
is linear
- There is a unique vector space structure on T_pS such that Φ_* is a linear isomorphism for each coordinate map $\Phi : D \to S$

Tangent Bundle

- Tangent Bundle of S is the disjoint union of tangent spaces

$$T_*S = \coprod_{p \in S} T_pS$$

- Every element of T_pS is a tangent vector v in the tangent space of a point $p \in S$
- If $p \neq q$, then $T_pS \cap T_qS = \emptyset$
- A vector field is a map $V : S \to T_*S$ such that $V(p) \in T_pS$ for every $p \in S$
- If $\Phi : D \to S$ is a coordinate map, there is a map

$$\begin{aligned} \Phi_* : D \times \widehat{\mathbb{R}}^2 &\to T_*S \\ (u, \hat{v}) &\mapsto \Phi_* v \in T_{\Phi(u)}S \end{aligned}$$

Cotangent Bundle

- Cotangent Bundle of S is the disjoint union of cotangent spaces
$$T^*S = \coprod_{p \in S} T_p^*S$$
- Every element of T_p^*S is a cotangent vector θ in the cotangent space of a point $p \in S$
- If $p \neq q$, then $T_p^*S \cap T_q^*S = \emptyset$
- A 1-form is a map $\theta : S \to T^*S$ such that $\theta(p) \in T_p^*S$ for every $p \in S$
- If $\Phi : D \to S$ is a coordinate map, there is a map
$$\Phi^* : T^*S \to D \times (\widehat{\mathbb{R}}^2)^*$$
$$(p, \theta) \mapsto (\Phi^{-1}(p), \Phi^*\theta)$$

Orientation of a Local 2-Manifold

- An orientation on a local 2-manifold is an orientation on T_pS that depends continuously on p
- A frame (v_1, v_2) of vector fields on a local 2-manifold S defines an orientation

Bundle of Symmetric 2-Tensors

- Recall that S^2V^* is the vector space of symmetric 2-tensors over a vector space V
- Bundle of symmetric 2-tensors over S is the disjoint union of symmetric 2-tensors over tangent spaces

$$S^2T^*S = \coprod_{p\in S} S^2T_p^*S$$

- Every element of $S^2T_p^*S$ is a a symmetric 2-tensor over the vector space T_pS
- If $p \neq q$, then $S^2T_p^*S \cap S^2T_q^*S = \emptyset$
- A symmetric 2-tensor field is a map $t : S \to S^2T^*S$ such that $t(p) \in S^2T_p^*S$ for every $p \in S$
- If $\Phi : D \to S$ is a coordinate map, there is a map

$$\begin{aligned}\Phi^* : S^2T^*S &\to D \times S^2(\widehat{\mathbb{R}}^2)^* \\ (p,t) &\mapsto (\Phi^{-1}(p), \Phi^*t),\end{aligned}$$

where $S^2(\widehat{\mathbb{R}}^2)^*$ is the space of symmetric 2-by-2 matrices

Dot Product on a Vector Space

- Recall that a dot product on a vector space is a positive definite symmetric 2-tensor
  - $g : V \times V \to \mathbb{R}$
  - $g(v_1 + v_2, w) = g(v_1, w) + g(v_2, w)$ and $g(v, w_1 + w_2) = g(v, w_1) + g(v, w_2)$
  - $g(cv, w) = g(v, cw) = cg(v, w)$
  - $g(w, v) = g(v, w)$
  - $g(v, v) > 0$ if $v \neq 0$
- Examples on $\widehat{\mathbb{R}}^2$
  - $g(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = x_1x_2 + y_1y_2$
  - $g(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = 2x_1x_2 + 3y_1y_2$
  - $g(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = x_1x_2 + x_1y_2 + x_2y_1 + 2y_1y_2$
- Non-examples on $\widehat{\mathbb{R}}^2$
  - $g(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = x_1x_2 - y_1y_2$
  - $g(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = x_1y_2 + x_2y_1$
  - $g(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = x_1x_2$
- Given a dot product g, there exists a basis (e_1, e_2) such that

$$g(e_1, e_1) = g(e_2, e_2) = 1 \text{ and } g(e_1, e_2) = g(e_2, g_1) = 0$$

Riemannian Metric on Local 2-Manifold

- A Riemannian metric g on a local 2-manifold S is a positive definite symmetric 2-tensor field
- If $p \in S$, then $g(p) \in S^2 T_p^*$ and

$$g_p(v, v) > 0, \text{ if } v \neq 0$$

- Each tangent space T_pS has its own dot product
- Example: Euclidean space $\mathbb{E}^2$
  - $T_p\mathbb{E}^2 = \mathbb{V}^2$
  - Dot product on $T_p\mathbb{E}^2$ is the dot product on $\mathbb{V}^2$
- Example: First fundamental form of a local surface $S \subset \mathbb{E}^3$
  - The dot product on each $T_pS \subset \mathbb{V}^3$ is the dot product on $\mathbb{V}^3$ restricted to T_pS

Hyperbolic Plane

- Given $p_0 \in \mathbb{E}^2$, let S be the open unit disk centered at p_0,

$$S = \{p \in \mathbb{E}^2 \ : \ |p - p_0| < 1\}$$

- Given $v, w \in \mathbb{V}^2$, let $v \cdot w$ be the Euclidean dot product
- The hyperbolic metric is the Riemannian metric g given by

$$g(p)(v, w) = \frac{4(v \cdot w)}{1 - |p - p_0|^2}$$

- Negatively curved analogue of the unit sphere
- No way to visualize it in Euclidean 3-space