MATH-UA 377 Differential Geometry: Moving frame on Surface in Euclidean Space Weingarten Map Second Fundamental Form

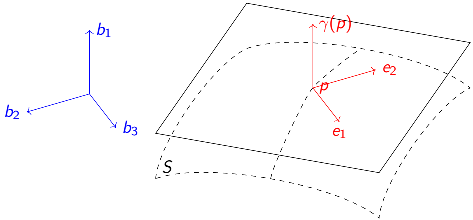
Deane Yang

Courant Institute of Mathematical Sciences
New York University

April 21, 2022

START RECORDING
LIVE TRANSCRIPTION

Oriented Surface in Oriented Euclidean 3-Space



- Let $\mathbb{E}^3$ be Euclidean space with an orientation (b_1, b_2, b_3)
- Let $S \subset \mathbb{E}^3$ be a parameterized surface with coordinate map $\Phi : D \to S$
- A parameterized surface is always orientable, because $(\partial_1\Phi, \partial_2\Phi)$ is an orientation on T_pS for each $p = \Phi(u^1, u^2)$
- An orientation on T_pS uniquely determines a unit normal $n(p)$ at each $p \in S$ and vice versa
- $(\partial_1\Phi, \partial_2\Phi)$ has positively orientation on T_pS if and only if $(\partial_1\Phi, \partial_2\Phi, n)$ is positively oriented on $\mathbb{V}^3$

Adapted Oriented Orthonormal Frame on $S \subset \mathbb{E}^3$

- (e_1, e_2, e_3) is an adapted oriented orthonormal frame on S if for each $p \in S$,
  - $(e_1(p), e_2(p), e_3(p))$ is positively oriented on $\mathbb{V}^3$
  - $(e_1(p), e_2(p))$ is positively oriented on T_pS
- For each $p \in S$, let $\mathcal{F}_pS$ be the set of all oriented orthonormal bases of T_pS
- For each frame $(e_1, e_2) \in \mathcal{F}_pS$, there is a unique oriented orthonormal basis (e_1, e_2, e_3) of $\mathbb{V}^3$
- The oriented orthonormal frame bundle of S is

$$\mathcal{F}_*S = \text{ disjoint union of } \mathcal{F}_pS \text{ for all } p \in S$$

- A moving orthonormal frame of S is a map

$$E = (e_1, e_2, e_3) : S \to \mathcal{F}_*S \text{ such that } E(p) \in \mathcal{F}_pS$$

Gauss Map of an Oriented Surface

- The Gauss map of an oriented surface $S \subset \mathbb{E}^3$ is defined to be

$$\begin{aligned} \gamma : S &\to \mathbb{V}^3 \\ p &\mapsto \text{ positively oriented unit normal to } T_pS \end{aligned}$$

- If (e_1, e_2) is an oriented orthonormal frame on S, then (e_1, e_2, γ) is an oriented orthonormal frame on $\mathbb{E}^3$

The Weingarten Map

- Given $p \in S$ and $v \in T_pS$, the directional derivative of $\gamma : S \to \mathbb{V}^3$ is

$$D_v\gamma(p) = \left.\frac{d}{dt}\right|_{t=0} \gamma(c(t)) \in \mathbb{V}^3 = \lim_{h\to 0} \frac{\gamma(c(t+h)) - \gamma(c(t))}{h},$$

  where $c(0) = p$ and $c'(0) = v$
- Since $\gamma \cdot \gamma = 1$,

$$0 = \left.\frac{d}{dt}\right|_{t=0} (\gamma(c(t))\cdot\gamma(c(t))) = 2\gamma(p)\cdot \left.\frac{d}{dt}\right|_{t=0} \gamma(c(t)) = 2\gamma\cdot D_v\gamma,$$

  which implies that $D_v\gamma(p) \in T_pS$
- Therefore, at each $p \in S$, we can define the differential of the Gauss map to be

$$\begin{aligned} d\gamma(p) : T_pS &\to T_pS \\ v &\mapsto D_v\gamma(p) \end{aligned}$$

- This is the Weingarten map

Example: Sphere of Radius R

- The sphere of radius R centered at $p_0 \in \mathbb{E}^3$ is

$$S = \{p \in \mathbb{E}^3 \ : \ (p - p_0) \cdot (p - p_0) = R^2\}$$

- Recall that $p - p_0$ is an outward normal vector to T_pS and therefore, the Gauss map is given by

$$\gamma(p) = \frac{p - p_0}{|p - p_0|} = \frac{p - p_0}{R}$$

  and the tangent space at p is

$$T_pS = \{v \in \mathbb{V}^3 \ : \ v \cdot \gamma(p) = 0\}$$

Weingarten Map of Sphere

- The Gauss map of the sphere of radius R centered at p_0 is given by

$$\gamma(p) = \frac{p - p_0}{R}, \ \ p \in S$$

- For each $v \in T_pS$, let c be a curve such that $c(0) = p$ and $c'(0) = v$,
- The directional derivative of the Gauss map is

$$\begin{aligned} \langle v, d\gamma(p) \rangle &= \left.\frac{d}{dt}\right|_{t=0} \gamma(c(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \frac{c(t) - p_0}{R} \\ &= \frac{c'(0)}{R} = \frac{v}{R} \end{aligned}$$

- Therefore, the Weingarten map at $p \in S$ is given by

$$\begin{aligned} d\gamma(p) : T_pS &\to T_pS \\ v &\mapsto \frac{v}{R} \end{aligned}$$

Weingarten Map of Ellipsoid in $\mathbb{R}^3$

- Given $a, b, c > 0$, let

$$S = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

- Given any $p = (x, y, z) \in S$, $v = \langle \dot{x}, \dot{y}, \dot{z} \rangle \in T_p S$, and curve $c : I \to S$ such that $c(0) = p$ and $c'(0) = v$,

$$0 = \frac{d}{dt}\bigg|_{t=0} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = 2\langle \dot{x}, \dot{y}, \dot{z} \rangle \cdot \left\langle \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\rangle$$

- Therefore, the Gauss map is

$$\gamma(x, y, z) = \frac{\left\langle \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\rangle}{\left| \left\langle \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\rangle \right|}$$

- The Weingarten map is given by

$$\langle v, d\gamma(p) \rangle = \frac{\left\langle \frac{\dot{x}}{a^2}, \frac{\dot{y}}{b^2}, \frac{\dot{z}}{c^2} \right\rangle}{\left| \left\langle \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\rangle \right|} - \frac{\left(\left\langle \frac{\dot{x}}{a^2}, \frac{\dot{y}}{b^2}, \frac{\dot{z}}{c^2} \right\rangle \cdot \left\langle \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\rangle \right) \left\langle \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\rangle}{\left| \left\langle \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\rangle \right|^3}$$

Weingarten Map At an Extreme Point of Ellipsoid

- At $p = (a, 0, 0)$,
$$\gamma(a, 0, 0) = \langle 1, 0, 0 \rangle$$
and therefore
$$T_{(a,0,0)}S = \{\langle 0, \dot{y}, \dot{z} \rangle\}$$
- The Weingarten map at $(a, 0, 0)$ is
$$\langle \langle 0, \dot{y}, \dot{z} \rangle, d\gamma \rangle = a \left\langle 0, \frac{\dot{y}}{b^2}, \frac{\dot{z}}{c^2} \right\rangle$$
- This can also be written as
$$\left\langle \begin{bmatrix} 0 \\ \dot{y} \\ \dot{z} \end{bmatrix}, d\gamma(a, 0, 0) \right\rangle = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{a}{b^2} & 0 \\ 0 & 0 & \frac{a}{c^2} \end{bmatrix} \begin{bmatrix} 0 \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

Second Fundamental Form of a Surface

- Let $\gamma : S \to \mathbb{V}^3$ be the Gauss map of S
- Let $d\gamma$ be the Weingarten map
- Recall that if $p \in S$ and $w \in T_pS$, then $\langle w, d\gamma(p)\rangle \in T_pS$
- Given two tangent vectors $v, w \in T_pS$,

$$\mathrm{II}(p)(v, w) = v \cdot \langle w, d\gamma(p)\rangle \in \mathbb{R}$$

- $\mathrm{II}(p)$ is a linear function of $v \in T_pS$ and a linear function of $w \in T_pS$
- $\mathrm{II}(p)$ is therefore a bilinear tensor
- The bilinear tensor field II is the second fundamental form
- We can therefore write

$$\langle v \otimes w, \mathrm{II}(p)\rangle \in \mathbb{R}$$

Differential of Gauss Map Using Moving Frame

- Let (e_1, e_2, e_3) be an orthonormal frame on S, where $e_3 = \gamma$ is the Gauss map
- Recall that one of the structure equations is

$$de_3 = e_1\omega_3^1 + e_2\omega_3^2$$

- Since (ω^1, ω^2) is a dual frame, there are functions H_{ij}, $1 \leq i, j \leq 2$, such that

$$\begin{aligned}
\omega_1^3 &= H_{11}\omega^1 + H_{12}\omega^2 \\
\omega_2^3 &= H_{21}\omega^1 + H_{22}\omega^2
\end{aligned}$$

- On the other hand, another structure equation is

$$\begin{aligned}
0 &= \omega_1^3 \wedge \omega^1 + \omega_2^3 \wedge \omega^2 \\
&= -(H_{11}\omega^1 + H_{12}\omega^2) \wedge \omega^1 - (H_{21}\omega^1 + H_{22}\omega^2) \wedge \omega^2 \\
&= (H_{12} - H_{21})\omega^1 \wedge \omega^2
\end{aligned}$$

Weingarten Map Using Moving Frame

- Let $v = v^1 e_1 + v^2 e_2$ and $w = w^1 e_1 + w^2 e_2$ be any two tangent vectors at p
- Therefore, then the Weingarten map is given by

$$
\begin{aligned}
\langle w, de_3 \rangle &= e_1 \langle w, \omega_3^1 \rangle + e_2 \langle w, \omega_3^2 \rangle \\
&= e_1 \langle w, \omega_1^3 \rangle e_2 \langle w, \omega_2^3 \rangle \\
&= e_1 \langle w^1 e_1 + w^2 e_2, H_{11}\omega^1 + H_{12}\omega^2 \rangle \\
&\quad + e_2 \langle w^1 e_1 + w^2 e_2, H_{21}\omega^1 + H_{22}\omega^2 \rangle \\
&= e_1 (H_{11} w^1 + H_{12} w^2) + e_2 (H_{21} w^1 + H_{22} w^2)
\end{aligned}
$$

Second Fundamental Form Using Moving Frame

- The second fundamental form is given by

$$\begin{aligned}\langle v \otimes w, \mathrm{II}(p)\rangle &= v \cdot \langle w, de_3\rangle \\ &= (v^1 e_1 + v^2 e_2) \\ &\quad \cdot (e_1(H_{11}w^1 + H_{12}w^2) + e_2(H_{21}w^1 + H_{22}w^2)) \\ &= H_{11}v^1 w^1 H_{12}(v^1 w^2 + v^2 w^1) H_{22} v^2 w^2 \\ &= \langle w \otimes v, \mathrm{II}(p)\rangle\end{aligned}$$

- The second fundamental form is a symmetric 2-tensor field
- Example: Sphere of radius R
  - The Weingarten map was

$$\langle v, d\gamma(p)\rangle = \frac{v}{R}$$

  - The second fundamental form is therefore

$$\langle v \otimes w, \mathrm{II}(p)\rangle = v \cdot \langle w, d\gamma(p)\rangle = v \cdot \frac{w}{R} = \frac{v \cdot w}{R}$$

Example: Graph of a Function

- Consider a parameterized surface given by a graph

$$\Phi : D \to \mathbb{R}^3$$
$$(x, y) \mapsto (x, y, f(x, y))$$

- A tangent frame is

$$(e_1, e_2) = \left(\frac{\partial_x \Phi}{|\partial_x \Phi|}, \frac{\partial_y \Phi}{|\partial_y \Phi|} \right)$$
$$= \left(\frac{\langle 1, 0, \partial_x f \rangle}{\sqrt{1 + \partial_x f^2}}, \frac{\langle 0, 1, \partial_y f \rangle}{\sqrt{1 + \partial_y f^2}} \right)$$

- A normal vector is

$$\langle -\partial_x f, -\partial_y f, 1 \rangle$$

- Therefore, an orthonormal frame is (e_1, e_2, e_3), where

$$e_3 = \frac{\langle -\partial_x f, -\partial_y f, 1 \rangle}{\sqrt{1 + f_x^2 + f_y^2}}$$