# MATH-UA 377 Differential Geometry: Moving Frame on Euclidean Space First Fundamental Form of a Surface Orthonormal Moving Frame on a Surface

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# START RECORDING
# LIVE TRANSCRIPTION

# Differential of Identity Map

- Let $I : \mathbb{A}^m \to \mathbb{A}^m$ be the identity map
- Given $v \in \mathbb{V}^m$ and a curve $c$ such that $c(0) = p \in \mathbb{A}^m$ and $c'(0) = v$, the directional derivative of $I$ at $p \in \mathbb{A}^m$ is

$$\begin{aligned} D_v I(p) &= \left.\frac{d}{dt}\right|_{t=0} I(c(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} c(t) \\ &= c'(0) \\ &= v, \end{aligned}$$

- On the other hand, we know that if $(e_1, \ldots, e_m)$ is a moving frame and $(\omega^1, \ldots, \omega^m)$, then

$$v = e_1 \langle \omega^1, v \rangle + \cdots + e_m \langle \omega^m, v \rangle$$

- So

$$dI = e_k \omega^k$$

# Structure Equations for Moving Frame on Affine Space

- There exist unique 1-forms $\omega^j_k$, where $1 \leq j, k \leq m$, such that the following equations hold:

$$\begin{aligned} dI &= e_k \omega^k \\ de_k &= e_j \omega^j_k \\ d\omega^j + \omega^j_k \wedge \omega^k &= 0 \\ d\omega^j_k + \omega^j_i \wedge \omega^i_k &= 0 \end{aligned}$$

- The 1-forms $\omega^j_k$ are called the connection 1-forms and measure the twisting of the frame $E$ as it moves around the domain $O$

# Matrix Form of Structure Equations

If we write

$$E = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$$

$$E^* = \begin{bmatrix} \omega_1 \\ \cdots \\ \omega_m \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} \omega_1^1 & \cdots & \omega_m^1 \\ \vdots & & \vdots \\ \omega_1^m & \cdots & \omega_m^m \end{bmatrix},$$

then the structure equations become

$$dI = EE^*$$

$$dE = E\Gamma$$

$$dE^* + \Gamma \wedge E^* = 0$$

$$d\Gamma + \Gamma \wedge \Gamma = 0,$$

# Orthonormal Moving Frame on Euclidean Space

- Let $\mathbb{A}^m = \mathbb{E}^m$, so $\mathbb{V}^m$ now has an inner product
- Let $E = (e_1, \ldots, e_m)$ be an orthonormal moving frame on $O \subset \mathbb{E}^m$
- Recall that this means $e_i \cdot e_j = \delta_{ij}$, which can be written as

$$E^t \cdot E = \begin{bmatrix} e_1 \\ \vdots \\ e_m \end{bmatrix} \cdot \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$$

$$= \begin{bmatrix} e_1 \cdot e_1 & \cdots & e_1 \cdot e_m \\ \vdots & & \vdots \\ e_m \cdot e_1 & \cdots & e_m \cdot e_m \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

# Dual Orthonormal Moving Frame

- Let $(\omega^1, \ldots, \omega^m)$ be the dual frame
- Since $e_j \cdot e_k = \delta_{jk}$,

$$\begin{aligned} 0 &= d(e_j \cdot e_k) \\ &= de_j \cdot e_k + e_j \cdot e_k \\ &= (e_i \omega_j^i) \cdot e_k + e_j \cdot (e_i \omega_k^i) \\ &= \omega_j^i (e_i \cdot e_k) + \omega_k^i (e_j \cdot e_i) \\ &= \omega_j^k + \omega_k^j \end{aligned}$$

- Therefore,

$$\omega_j^k + \omega_k^j = 0 \text{ or } \Gamma + \Gamma^t = 0$$

# Structure Equations of Orthonormal Moving Frame in $\mathbb{E}^m$

- Same structure equations as for $\mathbb{A}^m$ and two more:

$$dx = e_k \omega^k$$

$$de_k = e_j \omega^j_k$$

$$d\omega^j + \omega^j_k \wedge \omega^k = 0$$

$$d\omega^j_k + \omega^j_i \wedge \omega^i_k = 0$$

$$e_j \cdot e_k = \delta_{ij}$$

$$\omega^j_k + \omega^k_j = 0$$

- In matrix form:

$$dx = EE^*$$

$$dE = E\Gamma$$

$$dE^* = \Gamma \wedge E^*$$

$$d\Gamma + \Gamma \wedge \Gamma = 0$$

$$E^t \cdot E = I$$

$$\Gamma + \Gamma^t = 0$$

# Example: Constant Orthonormal Frame on $\mathbb{E}^m$

- Fix a point $p_0 \in \mathbb{E}^m$ and an orthonormal frame $E_0 = (e_1, \ldots, e_m)$ on $\mathbb{V}^m$
- We can define the constant moving frame $E$, where for any $p \in \mathbb{E}^m$

$$E(p) = E_0$$

- The dual frame $E_0^* = (\omega^1, \ldots, \omega^m)$ is also constant
- Therefore, $de_i = 0$, which implies $\omega_j^i = 0$, which implies $\Gamma = 0$
- The structure equations are therefore

$$\begin{aligned} dx &= E_0 E_0^* \\ dE &= 0 \\ dE^* &= 0 \\ \Gamma &= 0 \end{aligned}$$

# Structure Equations of Orthonormal Moving Frame in $\mathbb{E}^3$

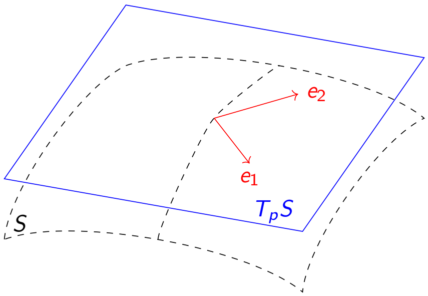
- $(e_1, e_2, e_3)$ orthonormal moving frame on $\mathbb{E}^3$
- $(\omega^1, \omega^2, \omega^3)$ dual frame
- Inner product on $\mathbb{V}^3$:

$$\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$$

- Structure equations

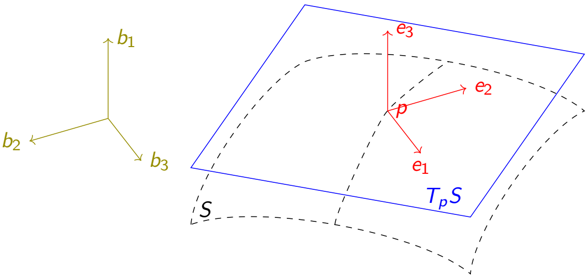
$$\begin{aligned} dx &= e_1\omega^1 + e_2\omega^2 + e_3\omega^3 \\ e_j \cdot e_k &= \delta_{ij} \\ \omega^j_k + \omega^k_j &= 0 \\ de_k &= e_j\omega^j_k \\ d\omega^j + \omega^j_k \wedge \omega^k + \omega^j_k \wedge \omega^k &= 0 \\ d\omega^j_k + \omega^j_i \wedge \omega^i_k &= 0 \end{aligned}$$

# First Fundamental Form of a Surface in Euclidean 3-Space



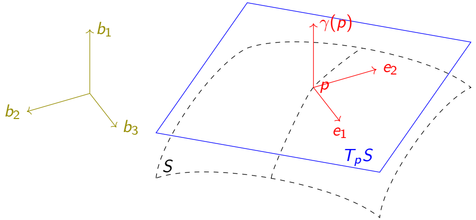

- Let $S \subset \mathbb{E}^3$ be a surface
- For each $p \in S$, the dot product on $\mathbb{V}$ restricted to $T_pS$ is a dot product on $T_pS$
- It therefore defines a symmetric 2-tensor field $g$, where for each $p \in S$, $g(p)$ is the dot product on $T_pS$
- $g$ is called the first fundamental form

# Orientation of a Surface $S \subset \mathbb{E}^3$



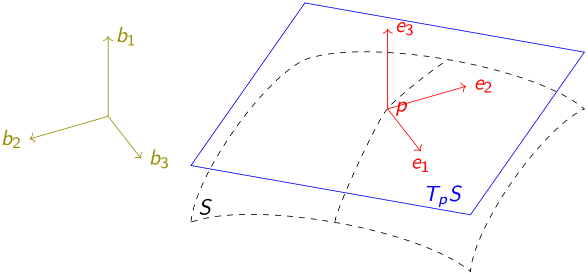

- Let $(b_1, b_2, b_3)$ be a positively oriented basis of $\mathbb{V}$
- Given a unit vector $e_3$ normal to $T_pS$, there is a unique orientation of $T_pS$ such that if $(e_1, e_2)$ is a positively oriented orthonormal basis of $T_pS$, then $(e_1, e_2, e_3)$ is a positively oriented basis of $\mathbb{V}$

# The Gauss Map of an Oriented Surface



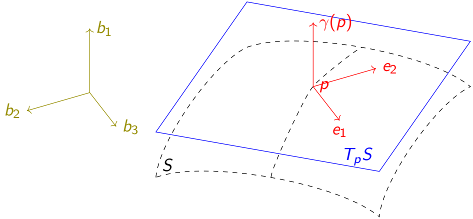

- Let $\mathbb{E}^3$ be Euclidean 3-space with a positively oriented basis $(b_1, b_2, b_3)$
- Let $S \subset \mathbb{E}^3$ be an oriented surface
  - Each $T_pS$ has an orientation, which depends continuously on $p \in S$
- At $p \in S$, let $(e_1, e_2)$ be a positively oriented basis of $T_pS$
- There is a unique vector $\gamma(p) \in \mathbb{V}^3$ such that
  - $\gamma(p)$ is a unit normal to $T_pS \subset \mathbb{V}^3$
  - $(e_1, e_2, \gamma(p))$ is positively oriented
- $\gamma$ is called the Gauss map of the oriented surface $S$

# Adapted Oriented Orthonormal Frame on $S \subset \mathbb{E}^3$




- An adapted oriented orthonormal frame on $S$ is an orthonormal frame $(e_1, e_2, e_3)$, where, for each $p \in S$,
  - $(e_1(p), e_2(p), e_3(p))$ is a positively oriented orthonormal basis of $\mathbb{V}^3$
  - $(e_1(p), e_2(p))$ is a positively oriented orthonormal basis of $T_pS$
  - In particular, $e_3(p) = \gamma(p)$ is the Gauss map

# Orthonormal Moving Frame and Dual Frame on Surface




- Let $\Phi : D \to S \cap O$ be a coordinate map
- Let $(\omega^1, \omega^2, \omega^3)$ be the dual frame of 1-forms
- We can pull a moving frame $(e_1, e_2, e_3)$ to $D$ using $\Phi$ to get maps

$$e_k \circ \Phi : D \to \mathbb{V},$$

  and 1-forms $(\Phi^*\omega^1, \Phi^*\omega^2, \Phi^*\omega^3)$
- $\Phi^*\omega^3 = 0$, because

$$\langle \Phi^*\omega^3, \partial_u \rangle = \langle \omega^3, \partial_u \Phi \rangle = 0$$

- $e_k$ will denote either $e_k : S \cap O \to \mathbb{V}$ or $e_k \circ \Phi : D \to \mathbb{V}$
- $\omega^k$ will denote either the 1-form $\omega^k$ on $S$ or its pullback $\Phi^*\omega^k$

# Structure Equations for Adapted Moving Frame on Surface

The structure equations for the moving frame and dual frame are

$$
\begin{aligned}
dl &= e_1\omega^1 + e_2\omega^2 \\
de_1 &= e_2\omega_1^2 + e_3\omega_1^3 \\
de_2 &= e_1\omega_2^1 + e_3\omega_1^3 \\
de_2 &= e_1\omega_2^1 + e_3\omega_2^3 \\
d\omega^j + \omega_k^j \wedge \omega^k &= 0,\ \ j = 1, 2 \\
\omega_1^3 \wedge \omega^1 + \omega_2^3 \wedge \omega^2 &= 0 \\
d\omega_k^j + \omega_i^j \wedge \omega_k^i &= 0, 1 \leq j, k \leq 3 \\
\omega_k^j + \omega_j^k &= 0,\ \ 1 \leq j, k \leq 3
\end{aligned}
$$