MATH-UA 377 Differential Geometry: Moving Frame on Affine Space

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LIVE TRANSCRIPTION

Differential of Coordinates on Affine Space

- Recall that given $p_0 \in \mathbb{A}^m$ and a basis $(\partial_1, \ldots, \partial_m)$ of $\mathbb{V}^m$, we have the coordinate map

$$\begin{aligned} \Phi : \widehat{\mathbb{R}}^m &\to \mathbb{A}^m \\ \langle x^1, \ldots, x^m \rangle &\mapsto p_0 + \partial_1 x^1 + \cdots + \partial_m x^m \\ &= Bx, \end{aligned}$$

  where

$$B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} \text{ and } x = \begin{bmatrix} x^1 \\ \vdots \\ x^m \end{bmatrix}$$

- Conversely, the inverse of Φ can be written as

$$\begin{aligned} \Phi^{-1} : \mathbb{A}^m &\to \mathbb{R}^m \\ p &\mapsto \langle x^1(p), \ldots, x^m(p) \rangle \end{aligned}$$

- Recall that the differentials $dx^1, \ldots, dx^m$ are constant 1-forms on $\mathbb{A}^m$ and $(dx^1, \ldots, dx^m)$ is the dual basis of $(\partial_1, \ldots, \partial_m)$

The Identity Map

- Let $B = (b_1, \ldots, b_m)$ be a basis of $\mathbb{V}$ and $B^* = (\beta^1, \ldots, b^m)$ be the dual basis
- Recall that for any $v = v^k b_k \in \mathbb{V}$, $\langle v, \beta^j \rangle = v^j$ and therefore

$$v = \langle v, \beta^k \rangle b_k$$

- In other words, the map

$$\begin{aligned} I : \mathbb{V} &\to \mathbb{V} \\ v &\mapsto \langle \beta^k, v \rangle b_k \end{aligned}$$

  is the identity map

Moving frame on Affine Space

- A moving frame on an open $O \subset \mathbb{A}^m$ consists of m vector fields
$$e_k : O \to \mathbb{V}^m,\ 1 \le k \le m,$$
such that, for each $x \in O$,
$$E(x) = (e_1(x), \ldots, e_m(x))$$
is a basis of $\mathbb{V}^m$
- The dual frame consists of 1-forms
$$\omega^k : O \to \mathbb{V}^*,\ 1 \le k \le m,$$
such that $E^*(x) = (\omega^1(x), \ldots, \omega^m(x))$ is the dual basis of $E(x)$
- For each $p \in \mathbb{A}^m$, the map
$$I : \mathbb{V} \to \mathbb{V}$$
$$v \mapsto e_k \langle \omega^k, v \rangle$$
is the identity map and therefore a constant map

Connection 1-forms

- The differential of $e_k : O \to \mathbb{V}$ at $x \in O$ is the map
$$de_k(x) : \mathbb{V} \to \mathbb{V},$$
where
$$\langle v, de_k(x) \rangle = \left.\frac{d}{dt}\right|_{t=0} e_k(x+tv) = \lim_{t \to 0} \frac{e_k(x+tv) - e_k(x)}{t} \in \mathbb{V}$$
- Since $(e_1(x), \dots, e_m(x))$ is a basis of $\mathbb{V}$, there are coefficients
$$a^1_k(x,v), \dots, a^m_k(x,v)$$
such that
$$\langle v, de_k(x) \rangle = a^1_k(x,v)e_1(x) + \cdots a^m_k(x,v)e_m(x)$$
- Since $\langle v, de_k(x) \rangle$ is a linear function of v, so is each $a^1_k(x,v)$
- Therefore, there are 1-forms ω^j_k such that
$$a^j_k(x,v) = \langle v, \omega^i_j(x) \rangle$$
and
$$de_k = e_j \omega^j_k$$

Fundamental Equations on Affine Space

- We now have two fundamental equations for a moving frame and its dual frame on an affine space:

$$I = e_k \omega^k$$
$$de_k = e_j \omega^j_k$$

- By taking the exterior derivative of these two equations, we will obtain equations involving the 1-forms only

First Structure Equation of Affine Space

- Since $I : \mathbb{V} \to \mathbb{V}$ is constant,

$$\begin{aligned} 0 = dI &= d(e_k \omega^k) \\ &= de_k \wedge \omega^k + e_k d\omega^k \\ &= e_j \omega^j_k \wedge \omega^k + e_j d\omega^j \\ &= e_j(\omega^j_k \wedge \omega^k + d\omega^j), \end{aligned}$$

- Since $(e_1, \ldots, e_m)$ is a basis, each coefficient has to be zero
- This implies the first structure equation

$$d\omega^j + \omega^j_k \wedge \omega^k = 0, \text{ for every } 1 \leq j \leq m$$

Second Structure Equation of Affine Space

- Treat each $e_k : O \to \mathbb{V}$ as if it is a scalar function
- Therefore,

$$
\begin{aligned}
0 = d(de_k) &= d(e_j \omega^j_k) \\
&= de_j \wedge \omega^j_k + e_j d\omega^j_k \\
&= e_i \omega^i_j \wedge \omega^j_k + e_j d\omega^j_k \\
&= e_j (d\omega^j_k + \omega^j_i \wedge \omega^i_k),
\end{aligned}
$$

- This implies the second structure equation

$$
d\omega^j_k + \omega^j_i \wedge \omega^i_k = 0, \text{ for every } 1 \leq j, k \leq m
$$

Structure Equations for Moving Frame on Affine Space

- Let $E = (e_1, \ldots, e_m)$ be a moving frame on an open $O \subset \mathbb{A}^m$ and $E^* = (\omega^1, \ldots, \omega^m)$ its dual frame
- There exist unique 1-forms ω^j_k, where $1 \leq j, k \leq m$, such that the following equations hold:

$$
\begin{aligned}
dx &= e_k \omega^k \\
de_k &= e_j \omega^j_k \\
d\omega^j + \omega^j_k \wedge \omega^k &= 0 \\
d\omega^j_k + \omega^j_i \wedge \omega^i_k &= 0
\end{aligned}
$$

- The 1-forms ω^j_k are called the connection 1-forms and measure the twisting of the frame as it moves around the domain O

Matrix Form of Structure Equations

If we write

$$
\begin{aligned}
E &= \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \\
E^* &= \begin{bmatrix} \omega_1 \\ \cdots \\ \omega_m \end{bmatrix} \\
\Gamma &= \begin{bmatrix} \omega_1^1 & \cdots & \omega_m^1 \\ \vdots & & \vdots \\ \omega_1^m & \cdots & \omega_m^m \end{bmatrix},
\end{aligned}
$$

then the structure equations become

$$
\begin{aligned}
I &= EE^* \\
dE &= E\Gamma \\
dE^* + \Gamma \wedge E^* &= 0 \\
d\Gamma + \Gamma \wedge \Gamma &= 0,
\end{aligned}
$$