MATH-UA 377 Differential Geometry: Orientation of a Rectangular Surface Integration of 2-form on an Oriented Rectangular Surface Stokes's Theorem on a Rectangular Surface Integration of 2-form on an Oriented Surface

Deane Yang

Courant Institute of Mathematical Sciences
New York University

April 12, 2022

START RECORDING
LIVE TRANSCRIPTION

Rectangular Surface in $\mathbb{A}^3$

- Let $\mathbb{A}^3$ be affine 3-space with tangent space $\mathbb{V}^3$
- A rectangular surface S is parameterized by a rectangle
- A rectangular surface consists of
  - Open and closed rectangles

$$R = (a, b) \times (c, d) \subset \overline{R} = [a, b] \times [c, d] \subset \mathbb{R}$$

  - A surjective C^1 map $\Phi : \overline{R} \to \overline{S} \subset \mathbb{A}^3$
  - The restriction of Φ to R is a coordinate map $\Phi : R \to S \cap O \subset \mathbb{A}^3$

Orientation on Parameterized Surface

- Orientation on a surface can be specified by a basis $(b_1(p), b_2(p)$ of T_p that depends continuously on $p \in S$
- A coordinate map $\Phi : D \to S$ is nondegenerate and bijective
- Therefore, $(b_1, b_2) = (\partial_1\Phi, \partial_2\Phi)$ is a basis of T_pS, for each $p = \Phi(x^1, x^2)$
- If $(\partial_1\Phi, \partial_2\Phi)$ is not the orientation we want, then we can switch the order of the input variables

Working Definition of Pullback of a 2-Form

- A 2-form Θ on an open set $O \subset \mathbb{R}^3$ can always be written as

$$\Theta = a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy,$$

  where a, b, c are scalar functions on O
- Consider a C^1 map $F : D \subset O$, where $D \subset \mathbb{R}^2$ is open
  - Does not have to be a coordinate map
- We can write $F(u, v) = (x(u, v), y(u, v), z(u, v))$
- The pullback of Θ by F is the 2-form on D

$$F^*\Theta = a\,dy \wedge dz + b\,dz \wedge dx + c\,dx \wedge dy, ,$$

  where

$$\begin{aligned} a &= a(x(u, v), y(u, v), z(u, v) \\ b &= b(x(u, v), y(u, v), z(u, v) \\ c &= c(x(u, v), y(u, v), z(u, v) \\ dx &= \partial_u x\,du + \partial_v x\,dv \\ dy &= \partial_u y\,du + \partial_v y\,dv \\ dz &= \partial_u z\,du + \partial_v z\,dv \end{aligned}$$

Integral of a 2-form on an Oriented Rectangular Surface S

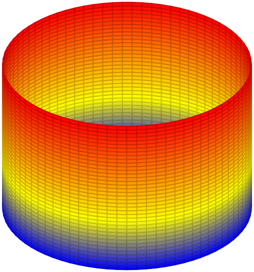
- Let $\Phi : R = [a, b] \times [c, d] \to S$ be a coordinate chart, where $(\partial_1 \Phi, \partial_2 \Phi)$ has the correct orientation
- The pullback $\Phi^* \Theta$ is a 2-form on R
- If (u^1, u^2) are coordinates on $\mathbb{R}^2$, then the pullback can be written as

$$\Phi^* \Theta = p(u^1, u^2)\, du^1 \wedge du^2$$

  where p is a scalar function on R
- The integral is defined to be

$$\begin{aligned}
\int_S \Theta &= \int_R \Phi^* \Theta \\
&= \int_{(u^1, u^2) \in r} p(u^1, u^2)\, du^1 \wedge du^2 \\
&= \int_{(u^1, u^2) \in D} p(u^1, u^2)\, du^1\, du^2 \\
&= \int_{u^1 = a}^{u^2 = b} \int_{u^2 = c}^{u^2 = d} p(u^1, u^2)\, du^2\, du^1
\end{aligned}$$

Example: Integral over cylinder



We want to compute

$$\int_C y\, dx \wedge dz,$$

where

$$C = \{x^2 + y^2 = \rho^2,\ 0 \leq z \leq h\}$$

Parameterization of Cylinder

- Use cylindrical coordinates

$$\Phi(u, v) = (\rho \cos u, \rho \sin u, v),$$

where $-\pi < u < \pi$ and $0 < v < h$

- Use orientation given by the basis $(\partial_u \Phi, \partial_v \Phi)$
- Therefore,

$$\begin{aligned} x &= \rho \cos u \\ y &= \rho \sin u \\ z &= v \\ dx &= -\rho \sin u \, du \\ dy &= \rho \cos u \, du \\ dz &= dv \end{aligned}$$

- The pullback of $y \, dx \wedge dz$ is

$$\begin{aligned} \Phi^*(y \, dx \wedge dz) &= (\rho \sin u)(-\rho \sin u) \wedge dv \\ &= -\rho^2 (\sin u)^2 \, du \wedge dv \end{aligned}$$

Integral Over Cylinder

- PUtting everything together, we get

$$
\begin{aligned}
\int_C y\,dx \wedge dz &= \int_R \Phi^*(y\,dx \wedge dz) \\
&= \int_R -\rho^2 (\sin u)^2\,du \wedge dv \\
&= \int_R -\rho^2 (\sin u)^2\,du\,dv \\
&= \int_{u=-\pi}^{u=\pi} \int_{v=0}^{v=h} -\rho^2 (\sin u)^2\,dv\,du \\
&= -\rho^2 \int_{u=-\pi}^{u=\pi} (\sin u)^2\,du \int_{v=0}^{v=h} dv \\
&= -\rho^2 h \int_{u=-\pi}^{u=\pi} (\sin u)^2\,du
\end{aligned}
$$

Stokes's Theorem for a Rectangular Surface

- Let ω be a 1-form on $O \subset \mathbb{R}^3$,

$$\omega = p\,dx + q\,dy + r\,dz$$

- Let S be a rectangular surface and $\Phi : R \to S$ be a coordinate map
- Stokes's Theorem says

$$\int_S d\omega = \int_{\partial S} \omega$$

Proof of Stokes's Theorem for a Rectangular Surface

- Key fact: If ω is a 1-form on an open set $O \subset \mathbb{R}^3$ and $F : D \to O$ is a C^1 map, then

$$F^*(d\omega) = d(F^*\omega)$$

- Recall Stokes's Theorem for a rectangle: If $R \subset \mathbb{R}^2$ is a rectangle and θ is a 1-form on $\overline{R}$, then

$$\int_R d\theta = \int_R \theta$$

- Therefore, if $\Phi : R \to S$ is a coordinate map for S, then

$$\int_S d\,\omega = \int_R \Phi^*(d\omega) = \int_R d(\Phi^*\omega) = \int_{\partial R} \Phi^*(\omega) = \int_{\partial S} \omega$$

- Crucial assumptions
  - $(\partial_1\Phi, \partial_2\Phi)$ is the desired orientation on S
  - The orientation of ∂S is consistent with the orientation of ∂R

Integration of a 2-Form over an Oriented Surface

- Idea: Chop S into rectangular surfaces $S_1, \ldots, S_N$, where

$$S = \overline{S}_1 \cup \cdots \cup \overline{S}_N$$

and $S_j \cap S_k = \emptyset$ for any $1 \le j, k \le N$

- If Θ is a 2-form on an open set $O \subset \mathbb{R}^3$ that contains S, then we define the integral of Θ over S to be

$$\int_S \Theta = \sum_{k=1}^{N} \int_{S_k} \Theta = \sum_{k=1}^{N} \int_{R_k} \Phi_k^* \Theta,$$

where each $\Phi_k : R_k \to S_k$ is a coordinate map

- Crucial assumption: The orientations used for the rectangular surfaces agrees with the orientation on S

Example: Integration over the Sphere

- Suppose $S = \{x^2 + y^2 + z^2 = \rho^2\}$ and we want to calculate
$$\int_S \Theta$$
- Use spherical coordinates
$$\begin{aligned}\Phi(\phi, \theta) &= (x, y, z)\\ &= (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi),\end{aligned}$$
where $0 < \phi < \pi$ and $0 < \theta < 2\pi$
- The integral is therefore
$$\int_S \Theta = \int_R \Phi^* \Theta,$$
where $R = (0, \pi) \times (0, 2\pi)$