MATH-UA 377 Differential Geometry: Double Integral of a function on a rectangle Integral of a 2-form on a rectangle Green's Theorem on a rectangle Integral of a 2-form on a rectangular surface

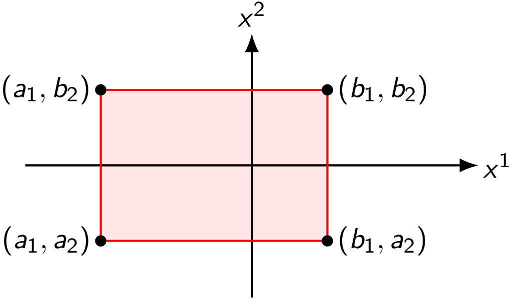
Deane Yang

Courant Institute of Mathematical Sciences
New York University

April 7, 2022

START RECORDING
LIVE TRANSCRIPTION

Integration over a rectangle in $\mathbb{R}^2$



- Consider the rectangular region

$$R = \{(x^1, x^2) \ : \ a_1 \leq x^1 \leq b_1 \text{ and } a_2 \leq x^2 \leq b_2\}$$
$$= [a_1, b_1] \times [a_1, b_2].$$

- The integral of a continuous function $f : R \to \mathbb{R}$ over R is defined to be

$$\int_R f(x)\, dx = \int_{x_1=a_1}^{x_1=b_1} \left(\int_{x^2=a_2}^{x_2=b_2} f(x^1, x^2)\, dx^2 \right) dx^1.$$

Fubini Theorem

- The order of itegration over a rectangle does not matter.
- $$\int_{x^1=a_1}^{x^1=b_1} \left(\int_{x^2=a_2}^{x^2=b_2} f(x^1, x^2)\, dx^2 \right) dx^1 = \int_{x^2=a_2}^{x^2=b_2} \left(\int_{x^1=a_1}^{x^1=b_1} f(x^1, x^2)\, dx^1 \right) dx^2.$$

Integration over a rectangular region in $\mathbb{R}^3$

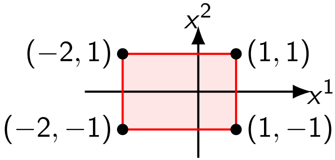
- Consider a 3-dimensional rectangular region,

$$R = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$$

- The integral of a continuous function $f : R \to \mathbb{R}$ over R is defined to be

$$\begin{aligned}
&\int_R f(x)\,dx \\
&= \int_{x_1=a_1}^{x_1=b_1} \left(\int_{x^2=a_2}^{x_2=b_2} \left(\int_{x^3=a_3}^{x_3=b_3} f(x^1, x^2, x^3)\,dx^3 \right) dx^2 \right) dx^1 \\
&= \int_{x_2=a_2}^{x_2=b_2} \left(\int_{x^1=a_1}^{x_1=b_1} \left(\int_{x^3=a_3}^{x_3=b_3} f(x^1, x^2, x^3)\,dx^3 \right) dx^1 \right) dx^2 \\
&= \int_{x_2=a_2}^{x_2=b_2} \left(\int_{x^1=a_3}^{x_3=b_3} \left(\int_{x^1=a_1}^{x_1=b_1} f(x^1, x^2, x^3)\,dx^1 \right) dx^3 \right) dx^2 \\
&= \cdots
\end{aligned}$$

Example



- Let $R = [-2,1] \times [-1,1]$
- Consider the integral

$$
\begin{aligned}
\int_R 4xy - 3y^2 \, dx \, dy &= \int_{x=-2}^{x=1} \int_{y=-1}^{y=1} 4xy - 3y^2 \, dy \, dx \\
&= \int_{x=-2}^{x=1} 2xy^2 - y^3 \Big|_{y=-1}^{y=1} \, dx \\
&= \int_{x=-2}^{x=1} (2x-1) - (2x+1) \, dx \\
&= \int_{x=-2}^{x=1} -2 \, dx \\
&= -6
\end{aligned}
$$

Oriented Parallelogram

- Given a point $p \in \mathbb{A}^2$ and a basis (v_1, v_2), we can define a parallelogram

$$P(p, v_1, v_2)\{p + t^1 v_1 + t^2 v_2 \;:\; 0 \leq t_1, t_2 \leq 1\}$$

- The corners are at $p, p + v_1, p + v_2, p + v_1 + v_2$
- An oriented parallelogram is the parallelogram P together with the orientation of (v_1, v_2)

Integral of a Constant 2-Form on an Oriented Parallelogram

- Let (∂_1, ∂_2) be the standard basis of $\widehat{\mathbb{R}}^2$
- Let (dx^1, dx^2) be the dual basis
- The integral of a constant 2-form

$$\Theta = c\, dx^1 \wedge dx^2$$

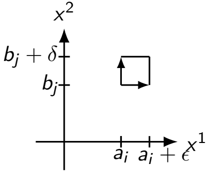
over an oriented parallelogram $P(p, v, w)$, where

$$v = v^1\partial_1 + v^2\partial_2, w = w^1\partial_1 + w^2\partial_2 \in \widehat{\mathbb{R}}^2$$

is defined to be

$$\begin{aligned}\int_R \Theta &= \langle v \otimes w, \Theta\rangle \\ &= \langle c\, dx^1 \wedge\ dx^2, v \otimes w\rangle \\ &= c(\langle dx^1, v\rangle\langle dx^2, w\rangle - \langle dx^2, v\rangle\langle dx^1, w\rangle \\ &= c(v^1w^2 - v^2w^1) \\ &= c(\text{signed area of } R)\end{aligned}$$

Integration of a 2-form over a small oriented rectangle

- A positively oriented rectangle is a parallelogram $P(p, a\partial_1, b\partial_2)$
- Chop R into small rectangles $P_{ij} = P(p_{ij}, a_i\partial_1, b_j\partial_2)$, $1 \leq i, j \leq N$.
- 

$$\int_{R_{ij}} \Theta \simeq \langle \Theta(a_i, b_j), (\epsilon\partial_1) \otimes (\delta\partial_2) \rangle$$
$$= \langle f(a_i, b_j) dx^1 \wedge dx^2), \epsilon\delta\partial_1 \otimes \partial_2 \rangle$$
$$= f(a_i, b_j)\epsilon\delta \langle dx^1 \wedge dx^2 \rangle$$
$$= f(a_i, b_j)(\text{area of } R_{ij})$$

Integration of a 2-form over an oriented rectangle

- The integral of Θ over a rectangular region R is defined to be

$$\begin{aligned}
\int_R \Theta &= \lim_{N\to\infty} \sum_{1\le i,j\le N} \int_{R_{ij}} \Theta \\
&= \lim_{N\to\infty} \sum_{1\le i,j\le N} \int_{R_{ij}} f(a_i, b_j)\,\mathrm{area}(R_{ij}) \\
&= \int_{x^1=p^1}^{x^1=p^1+a} \int_{x^2=p^2}^{x^2=p^2+b} f(x^1, x^2)\, dx^2\, dx^1
\end{aligned}$$

Integration of 2-form on a rectangle in practice

- Let $\Theta = f\, dx^1 \wedge dx^2$.
- Let $R = [a_1, b_1] \times [a_2, b_2]$.
- $$\begin{aligned}\int_R \Theta &= \int_{x^1=a_1}^{x^1=b_1} \left(\int_{x^2=a_2}^{x^2=b_2} f(x^1, x^2)\, dx^2 \right) dx^1 \\ &= \int_{x^2=a_2}^{x^2=b_2} \left(\int_{x^1=a_1}^{x^1=b_1} f(x^1, x^2)\, dx^1 \right) dx^2\end{aligned}$$
- Higher dimensional integral over a rectangular region (Cartesian product of intervals) is defined similarly

Order matters!

- Before you do the integration, you must write the m-form with the $dx^1, \ldots, dx^m$ in the correct order.
- $$\int_R f(x^1,x^2)\,dx^1 \wedge dx^2 = \int_{x^1=a_1}^{x^1=b_1}\left(\int_{x^2=a_2}^{x^2=b_2} f(x^1,x^2)\,dx^2\right)\,dx^1 = \int_{x^2=a_2}^{x^2=b_2}\left(\int_{x^1=a_1}^{x^1=b_1} f(x^1,x^2)\,dx^1\right)\,dx^2$$
- But
$$\int_R f(x^1,x^2)\,dx^2 \wedge dx^1 = -\int_R f(x^1,x^2)\,dx^1 \wedge dx^2 = -\int_{x^1=a_1}^{x^1=b_1}\left(\int_{x^2=a_2}^{x^2=b_2} f(x^1,x^2)\,dx^2\right)\,dx^1 = -\int_{x^2=a_2}^{x^2=b_2}\left(\int_{x^1=a_1}^{x^1=b_1} f(x^1,x^2)\,dx^1\right)\,dx^2$$

Example

- Suppose $R = [-4, 1] \times [0, 4]$.
- $$\begin{aligned}
\int_R (x+y)\, dy \wedge dx &= -\int_R (x+y)\, dx \wedge dy \\
&= -\int_{x=-4}^{x=1} \left(\int_{y=0}^{y=4} x + y\, dy \right) dx \\
&= -\int_{x=-4}^{x=1} \left(xy + \frac{y^2}{2} \Big|_{y=0}^{y=4} \right) dx \\
&= -\int_{x=-4}^{x=1} 4x + 8\, dx \\
&= -\left(2x^2 + 8x\right)\Big|_{x=-4}^{x=1} \\
&= -((2+8) - (32-32)) = -10
\end{aligned}$$

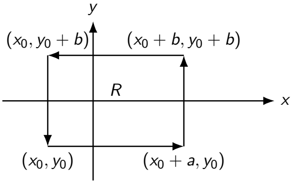
Example

- Suppose $R = [0,1] \times [0,4]$.
- Suppose $\theta = x\,dx + y\,dy$ and $\phi = y\,dx - x\,dy$.

$$\begin{aligned}
\int_R \theta \wedge \phi &= \int_R (x\,dx + y\,dy) \wedge (y\,dx - x\,dy) \\
&= \int_R -x^2\,dx \wedge dy + y^2\,dy \wedge dx \\
&= -\int_R (x^2 + y^2)\,dx \wedge dy \\
&= -\int_{y=0}^{y=4} \left(\int_{x=0}^{x=1} (x^2 + y^2)\,dx \right) dy \\
&= -\int_{y=0}^{y=4} \left(\frac{x^3}{3} + y^2 x \Big|_{x=0}^{x=1} \right) dy \\
&= -\int_{y=0}^{y=4} \frac{1}{3} + y^2\,dy = -\left(\frac{y}{3} + \frac{y^3}{3} \Big|_{y=0}^{y=4} \right) = \frac{4}{3} + \frac{64}{3} \\
&= -\frac{68}{3}
\end{aligned}$$

Orientation of the boundary of a rectangle

- Consider a rectangle with standard orientation



- The oriented boundary of R is the boundary with the orientation where R lies to the left of the curve

Line integral of 1-form around boundary of rectangle

The line integral of $\theta = P(x,y)\,dx + Q(x,y)\,dy$ along the oriented boundary ∂R is

$$\int_{\partial R} \theta = \int_{x=x_0}^{x=x_0+a} P(x,y_0)\,dx + \int_{y=y_0}^{y=y_0+b} Q(x_0+a,y)\,dy$$
$$\int_{x=x_0+a}^{x=x_0} P(x,y_0+b) + \int_{y=y_0+b}^{y=y_0} Q(x_0,y)\,dy$$

Fundamental theorem of calculus on a rectangle

$$\begin{aligned}
\int_{\partial R} \theta &= \int_{x=x_0}^{x=x_0+a} P(x, y_0)\, dx + \int_{y=y_0}^{y=y_0+b} Q(x_0+a, y)\, dy \\
&\quad + \int_{x=x_0+a}^{x=x_0} P(x, y_0+b) + \int_{y=y_0+b}^{y=y_0} Q(x_0, y)\, dy \\
&= \int_{y=y_0}^{y=y_0+b} Q(x_0+b, y) - Q(x_0, y)\, dy \\
&\quad - \int_{x=x_0}^{x=x+0+a} P(x, y_0+b) - P(x, y_0)\, dx \\
&= \int_{y=y_0}^{y=y_0+b} \int_{x=x_0}^{x=x_0+b} \partial_x Q(x, y)\, dx\, dy \\
&\quad - \int_{x=x_0}^{x=x_0+a} \int_{y=y_0}^{y=y_0+b} \partial_y P(x, y)\, dy\, dx \\
&= \int_R \partial_x Q - \partial_y P\, dx\, dy = \int_R \partial_x Q - \partial_y P\, dx \wedge dy
\end{aligned}$$

Exterior derivative of a 1-form

- Given a 1-form

$$\theta = P(x,y)\,dx + Q(x,y)\,dy$$

on an open domain $D \subset \mathbb{R}^2$, its exterior derivative is defined to be the 2-form

$$d\theta = (\partial_x Q - \partial_y P)\,dx \wedge dy$$

- Another way to write the definition is

$$d\theta = dP \wedge dx + dQ \wedge dy$$

because

$$\begin{aligned} dP \wedge dx + dQ \wedge dy &= (\partial_x P\,dx + \partial_y P\,dy) \wedge dx \\ &\quad + (\partial_x Q\,dx + \partial_y Q\,dy) \wedge dy \\ &= \partial_y P\,dy \wedge dx + \partial_x Q\,dx \wedge dy \\ &= (\partial_x Q - \partial_y P)\,dx \wedge dy \end{aligned}$$

Fundamental theorem of calculus on a rectangle

- Let ∂R be the boundary of R, oriented so that R lies to the left of the boundary.
- Let θ be a 1-form on an open set containing R.
- Since the formulas match, we get the Fundamental Theorem of Calculus on a rectange:

$$\int_R d\theta = \int_{\partial R} \theta.$$

- If $\theta = P\,dx + Q\,dy$, then this is

$$\int_R (Q_x - P_y)\,dx \wedge dy = \int_{\partial R} P\,dx + Q\,dy,$$

which is also known as Green's Theorem for a rectangle

Integration of 2-form over a surface parameterized by a rectangle

- Let $\Phi : D \to \mathbb{A}^m$ be a coordinate map, where D is an open subset of $\mathbb{R}^2$
- Let $S = \Phi(D)$ be the surface parameterized by Φ
- Let $R \subset D$ be a rectangle
- If Θ is a 2-form on D, then

$$\int_{\Phi(R)} \Theta = \int_R \Phi^* \Theta.$$

Integration using Polar Coordinates

- Let $D = (0, \infty) \times (-\pi, \pi) \subset \mathbb{R}^2$
- Let $\Phi : D \to \mathbb{R}^2$ be the coordinate map given by polar coordinates:

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta) \text{ or } (x, y) = (r\cos\theta, r\sin\theta)$$

- If $\Theta = f(x, y)\, dx \wedge dy$, then

$$\Phi^*\Theta = f(r\cos\theta, r\sin\theta) d(r\cos\theta) \wedge d(r\sin\theta) = f(r\cos\theta, r\sin\theta) r\, dr \wedge$$

- Therefore, If $R \subset (0, \infty) \times (-\pi, \pi)$ is a rectangle and $S = \Phi(R)$, then

$$\begin{aligned}\int_S \Theta &= \int_S f(x, y)\, dx \wedge dy \\ &= \int_R \Phi^*\Theta \\ &= \int_R f(r\cos\theta, r\sin\theta) r\, dr \wedge d\theta\end{aligned}$$

Example

- Iff $R = [r_1, r_2] \times [\theta_, \theta_2] \subset D, S = \Phi(R)$ and $\Theta = (x^2 + y^2)\, dx \wedge dy$, then

$$\begin{aligned}
\int_S \Theta &= \int_R \Phi^* \Theta \\
&= \int_{r=r_1}^{r=r_2} \int_{\theta=\theta_1}^{\theta=\theta_2} r^2 (r\, dr \wedge d\theta) \\
&= \int_{r=r_1}^{r=r_2} \int_{\theta=\theta_1}^{\theta=\theta_2} r^3\, dr\, d\theta \\
&= \int_{r=r_1}^{r=r_2} r^3\, dr \int_{\theta=\theta_1}^{\theta=\theta_2} d\theta \\
&= \frac{1}{4}(r_2^4 - r_1^4)(\theta_2 - \theta_1)
\end{aligned}$$