MATH-UA 377 Differential Geometry: Coordinate Vector Fields and 1-Forms Exterior Derivative of a Function Pullback of a 1-form to Parameterized Curve Line Integral of a 1-form Fundamental Theorem of Line Integrals

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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START RECORDING
LIVE TRANSCRIPTION

Tangent Space of a surface S

- Consider a surface $S \subset \mathbb{A}^m$, where $m = 2$ or 3
- The tangent space T_pS at each point p is the vector space of all possible velocity vectors at p.
  - $v \in T_pS$ if and only if there is a curve c such that $c(0) = p$ and $c'(0) = p$
- The tangent bundle is the disjoint union of all tangent spaces,

$$T_*S = \coprod_{p \in S} T_pS$$

  If $v \in T_*S$, then there is a unique point $p \in S$ such that $v \in T_pS$
- A map $V : S \to T_*S$ such that $V(p) \in T_pS$
- Given a coordinate map $\Phi : D \to S \subset \mathbb{A}^m$, there is a frame (∂_1, ∂_2)
  - A frame is an ordered pair of vector fields, (V_1, V_2), such that for any $p \in S$, $(V_1(p), V_2(p))$ is a basis of T_pS

Cotangent space

- The cotangent space at each $p \in S$ is $T_p^* S = (T_p S)^*$
- The cotangent bundle is the disjoint union of all cotangent spaces,

  $$T^* S = \coprod_{p \in S} T_p^* S$$

  If $\theta \in T^* S$, then there is a unique point $p \in S$ such that $\theta \in T_p^* S$
- A 1-form is a map $\theta : S \to T^* S$ such that $\theta(p) \in T_p^* S$
- Given a coordinate map $\Phi : D \to S$, the inverse map Φ^{-1} defines coordinate functions $x^i : \Phi(D) \to \mathbb{R}$
- The differential of the coordinate functions define the coordinate 1-forms (dx^1, dx^2)

Pullback of a Function

- Consider a function $f(x^1, x^2)$ written with respect to coordinates (x^1, x^2), like

$$f(x^1, x^2) = (x^1)^2 + (x^2)^2$$

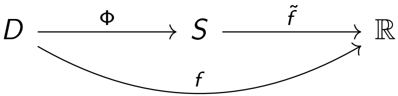
- This defines a functionn $\tilde{f} = f \circ \Phi^{-1} : S \to \mathbb{R}$, where

$$\tilde{f}(p) = f(x^1(p), x^2(p))$$

- Conversely, given a function $\tilde{f} : S \to \mathbb{R}$, we can define a function $f = \tilde{f} \circ \Phi : D \to \mathbb{R}$, where

$$f(x^1, x^2) = \tilde{f}(\Phi(x^1, x^2))$$

- We call f the pullback of $\tilde{f}$ by the map Φ



Differential of Function Using Coordinates

- Given a function $f(x^1, x^2)$, its differential is

$$df = \partial_1 f \, dx^1 + \partial_2 f \, dx^2$$

- Here, $(x^1, x^2) : \Phi(D) \to S$ and $f(x^1, x^2)$ really means $f(x^1(p), x^2(p))$
- So this formula shows how to write df in terms of the coordinate functions (x^1, x^2) and their differentials
- Using this, we get all the standard rules of differentiation
- Sum: $d(f + g) = df + dg$
- Constant factor: $d(cf) = c\, df$
- Product: $d(fg) = g\, df + f\, dg$
- Quotient: $d\left(\frac{f}{g}\right) = \frac{g\, df - f\, dg}{g^2}$
- Chain:

$$\begin{aligned} d(u \circ f) &= (u' \circ f) df \\ &= u'(f(x^1, x^2)) \left(\partial_1 f(x^1, x^2)\, dx^1 + \partial_2 f(x^1, x^2)\, dx^2\right) \end{aligned}$$

Examples

- 1-forms

$$\alpha = x\,dx + y\,dy$$

$$\theta = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

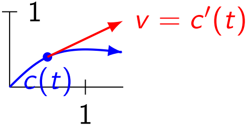
$$= \left(\frac{-y}{x^2 + y^2}\right)\,dx + \left(\frac{x}{x^2 + y^2}\right)\,dy$$

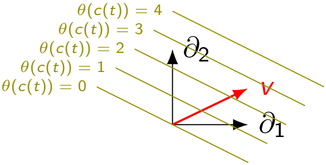
- Exterior derivative of a function

$$d(xy) = \partial_x(xy)\,dx + \partial_y(xy)\,dy = y\,dx + x\,dy$$

$$d(u^2 + v^2) = \partial_u(u^2 + v^2)\,du + \partial_v(u^2 + v^2)\,dv = 2u\,du + 2v\,dv$$

Line Integral of 1-form Along Oriented Curve





- Suppose

$$c(t) = \left(t, \frac{t}{1+t^2}\right), \; t \in [0, 2]$$

and

$$\theta = x^2\, dx^1 - x^1\, dx^2$$

- Calculate

$$\int_c \theta$$

Pullback of a 1-Form to a Parameterized Curve

- Given a 1-form θ on S and a curve $c : I \to S$, we define the pullback of θ by c to be the 1-form $c^*\theta$ on I, where

$$c^*\theta = \langle \theta(c(t)), c'(t) \rangle \, dt$$

- If

$$\theta = a\,dx + b\,dy,$$

then $c^*\theta$ is the 1-form on I you get if you replace x and y by their parameterizations and calculate dx and dy using this parameterization

$$\begin{aligned} c^*\theta &= a(x,y)\,dx + b(x,y)\,dy \\ &= a(x(t),y(t))\,x'(t)\,dt + b(x(t),y(t))\,y'(t)\,dt \\ &= (ax' + by')\,dt \end{aligned}$$

- Example: If $c(t) = (1+t^2, 2t)$, then

$$x = 1 + t^2 \text{ and } y = 2t$$

and therefore,

$$c^*(y^2\,dx + x\,y) = (2t)^2(2t\,dt) + (1+t^2)\,2\,dt = (8t^3 + 2t^2 + 2)\,dt$$

Line Integral of a 1-Form Along an Oriented Curve

- Consider an oriented interval $I = [a, b]$, a parameterized curve $c : I \to S$ and a 1-form θ
- The line integral of θ along the oriented curve is defined to be

$$\int_I c = \int_{t=a}^{t=b} c^* \theta$$

- By the chain rule (also known as substitution), the value of this integral is independent of the parameterization

Example

- Calculate
$$\int_C x^2\, dx^1 - x^1\, dx^2,$$
where C is the oriented curve with a parameterization
$$c(t) = \left(t, \frac{t}{1+t^2}\right), \; 0 \le t \le 1$$
- Write the curve as
$$x^1 = t \qquad\qquad x^2 = \frac{t}{1+t^2}$$
- Their differentials are
$$dx^1 = dt \qquad\qquad dx^2 = \frac{1-t^2}{(1+t^2)^2}\, dt$$
- Therefore,
$$\begin{aligned}\theta &= x^2\, dx^1 - x^1\, dx^2 \\ &= \frac{t}{1+t^2}\, dt - t\left(\frac{1-t^2}{(1+t^2)^2}\right)\, dt\end{aligned}$$
$$2t^3\, dt$$

Example of Line Integral

- If $c(t) = (x(t), y(t))$, then, along the curve,

$$dx = x'(t)\,dt \qquad\qquad dy = y'(t)\,dt$$

- If $\theta = a_1(x, y)\,dx + a_2(x, y)\,dy$, then, along the curve,

$$\theta = a_1(x(t), y(t))x'(t)\,dt + a_2(x(t), y(t))y'(t)\,dt$$

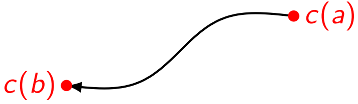
- Therefore,

$$\int_c \theta = \int_{t=a}^{t=b} (a_1(x, y)x'(t) + a_2(x, y)y'(t))\,dt$$

Line Integral is independent of Parameterization

- Suppose we reparameterize $c : [a, b] \to \mathbb{R}^2$ by a new parameter s,
$$\tilde{c}(s) = c(t(s)), \ \alpha \le s \le \beta,$$
where $t(\alpha) = a$ and $t(\beta) = b$
- By the chain rule,
$$\tilde{c}'(s) = c'(t(s))t'(s)$$
- The line integral of the new parameterized curve is
$$\begin{aligned}
\int_{\tilde{c}} \theta &= \int_{s=\alpha}^{s=\beta} \langle \theta(\tilde{c}(s)), \tilde{c}'(s) \rangle \, ds \\
&= \int_{s=\alpha}^{s=\beta} \langle \theta(c(t(s))), c'(t(s))t'(s) \rangle \, ds \\
&= \int_{s=\alpha}^{s=\beta} \langle \theta(c(t(s))), c'(t(s)) \rangle t'(s) \, ds \\
&= \int_{t=a}^{t=b} \langle \theta(c(t)), c'(t) \rangle dt = \int_{c} \theta
\end{aligned}$$

Line Integral of 1-form Along Oriented Curve

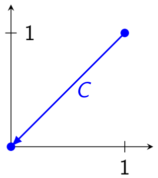


- In the definition of a line integral

$$\int_c \theta = \int_{t=a}^{t=b} \langle \theta(c(t)), c'(t) \rangle \, dt,$$

  we do NOT have to assume that $a \leq b$
- If $c : [a, b]$ is a parameterization of an oriented curve, we can set a to be the starting value of the parameter and b to be the ending value of the parameter
- In other words, $c(a)$ is the start of the curve and $c(b)$ is the end of the curve

Example



- Suppose we want to compute $\int_C (x^1+1)\,dx^2 + (x^2-1)\,dx^1$
- We can parameterize C by

$$c(t) = (t,t),\ 0 \le t \le 1,$$

where the start of C is $c(1)$ and the end of C is $c(0)$

- Then

$$\int_C x^1\,dx^2 - x^2\,dx^1 = \int_{t=1}^{t=0} (t+1)\,dt + (t-1)\,dt$$
$$= \int_{t=1}^{t=0} 2t\,dt = t^2\Big|_{t=1}^{t=0} = -1$$

Fundamental Theorem of Line Integrals

- Suppose $\theta = df$, where $f : S \to \mathbb{R}$
- Along a curve $c : [a, b] \to S$,

$$\begin{aligned}
df &= \partial_1 f\, dx^1 + \partial_2 f\, dx^2 \\
&= (\partial_1 f(x^1, x^2)(x^1)'(t) + \partial_2 f(x^1, x^2)(x^2)'(t))\, dt \\
&= \left(\frac{d}{dt} f(x^1(t), x^2(t)) \right)\, dt \\
&= (f \circ c)'(t)\, dt
\end{aligned}$$

- Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_c df &= \int_{t=a}^{t=b} \left(\frac{d}{dt} f(x^1(t), x^2(t)) \right)\, dt \\
&= f(x^1(b), x^2(b)) - f(x^1(a), x^2(s)) \\
&= f(c(b)) - f(c(a))
\end{aligned}$$

Example

- Let $c : [a, b] \to \mathbb{R}^2$ be a curve that starts at $(-1, 2)$ and ends at $(5, 7)$
- Suppose we want to calculate

$$\int_c x^2\, dx^1 + x^1\, dx^2$$

- Since $d(x^1 x^2) = x^2\, dx^1 + x^1\, dx^2$,

$$\begin{aligned}
\int_c x^2\, dx^1 + x^1\, dx^2 &= \int_c d(x^1 x^2) \\
&= x^1(b)x^2(b) - x^1(a)x^2(a) \\
&= 5(7) - (-1)(2) \\
&= 37
\end{aligned}$$

Consequences

- The fundamental theorem of line integrals says

$$\int_c df = f(\text{end of } c) - f(\text{start of } c)$$

- If c_1 and c_2 are two curves that start at the same point and end at the same point, then

$$\int_{c_1} df = \int_{c_2} df$$

- If c is a closed curve (i.e., its start and end points are the same), then

$$\int_c df = 0$$