MATH-UA 377 Differential Geometry: Coordinate Vector Fields and 1-Forms Exterior Derivative of a Function in Coordinates Line Integral

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START RECORDING
LIVE TRANSCRIPTION

Coordinates on a Parameterized Surface

- Let $S \subset \mathbb{A}^m$, where $m = 2$ or 3, be a parameterized surface and
$$\Phi : D \to S$$
be a coordinate map, where D is an open subset of $\mathbb{R}^2$
- The inverse map
$$\Phi^{-1} : S \to D$$
consists of scalar functions
$$x^1 : S \to \mathbb{R} \text{ and } x^2 : S \to \mathbb{R},$$
where for each $p \in D$,
$$\Phi(x^1(p), x^2(p)) = p$$
or, equivalently,
$$x^1(\Phi(y^1, y^2)) = y^1, \text{ i.e., } x^1\Phi(s, t) = s$$

Examples

- Standard coordinates on $\mathbb{R}^2$
  - Parameter domain: $D = \mathbb{R}^2$
  - Surface: $S = \mathbb{R}^2$
  - Coordinate map is identity map, i.e.,

$$\Phi(x^1, x^2) = (x^1, x^2)$$

- Polar coordinates on $\mathbb{R}^2$
  - Parameter domain: $D = (0, \infty) \times (-\pi, \pi)$
  - Surface: $S = \mathbb{R}^2 \backslash \{(0, y) \ : \ y \leq 0\}$
  - Coordinate map: $\Phi(r, \theta) = (r \cos \theta, r \sin \theta)$

Coordinate Vector fields on Surface

- Let $T_pS \subset \mathbb{V}^m$ be the tangent space of S at $p \in S$
  - If $m = n2$, then $T_pS = \mathbb{V}^2$
- A vector field on S is a map $V : S \to \mathbb{V}^m$ such that $V(p) \in T_pS$
- Given any point $p = \Phi(x^1, x^2) \in S$, we can define the curves

$$c_1(t) = \Phi(x^1 + t, x^2) \text{ and } c_2(t) = \Phi(x^1, x^2 + t)$$

- $c_1(0) = c_2(0) = \Phi(x^1, x^2) = p$
- Velocity of each curve at p is

$$c_1'(0) = \partial_1\Phi(x^1, x^2) \text{ and } c_2'(0) = \partial_2\Phi(x^1, x^2)$$

- Since Φ is nondegenerate, these two vectors are a basis of T_pS

Coordinate Vector Fields on Surface

- Coordinate vector fields of a coordinate map $\Phi(x^1, x^2)$ are ∂_1 and ∂_2, where for each $p \in S$,

$$\partial_1(p) = \partial_1 \Phi(x^1(p), x^2(p)) \in T_p S$$
$$\partial_2(p) = \partial_2 \Phi(x^1(p), x^2(p)) \in T_p S$$

Velocity of a Curve With Respect to Coordinates

- Consider a curve $c : I \to S$ such that $c(0) = p$ and $c'(0) = v$
- There are functions $x^1 : I \to \mathbb{R}$ and $x^2 : I \to \mathbb{R}$ such that

$$c(t) = \Phi(x^1(t), x^2(t)) \in S$$

- On one hand, since (∂_1, ∂_2) is a basis of T_pS, the velocity of c at $t = 0$ can be written as

$$c'(0) = v = v^1\partial_1 + v^2\partial_2$$

- On the other hand, by the chain rule,

$$\begin{aligned} c'(0) &= \partial_1\Phi(x^1, x^2)\dot{x}^1(0) = \partial_2\Phi(x^1, x^2)\dot{x}^2(0) \\ &= \dot{x}^1\partial_1 + \dot{x}^2\partial_2 \end{aligned}$$

- Therefore, if $c(t) = \Phi(x^1(t), x^2(t))$, then

$$\dot{x}^1(0) = v^1 \text{ and } \dot{x}^2(0) = v^2$$

Coordinate 1-forms on a Surface

- Given a function $f : S \to \mathbb{R}$, its exterior derivative is a 1-form df, where for each $p \in S$ and $v \in T_pS$,

$$\langle v, df(p)\rangle = \left.\frac{d}{dt}\right|_{t=0} f(c(t)),$$

where c is a curve such that $c(0) = p$ and $c'(0) = v$

- Therefore, if dx^1 is the differential of the coordinate function $x^1 : S \to \mathbb{R}$ and $v = v^1\partial_1 + v^2\partial_2$, then

$$\langle v, dx^1(p)\rangle = \left.\frac{d}{dt}\right|_{t=0} x^1(\Phi(x^1(t), x^2(t)) = \left.\frac{d}{dt}\right|_{t=0} x^1(t) = \dot{x}^1 = v^1$$

- It follows that for each $p \in S$,

$$(dx^1(p), dx^2(p)) \text{ is the basis of } T_p^*S$$

dual to the basis

$$(\partial_1(p), \partial_2(p)) \text{ of } T_pS$$

Pullback of a Function

- Consider a function $f(x^1, x^2)$ written with respect to coordinates (x^1, x^2), like

$$f(x^1, x^2) = (x^1)^2 + (x^2)^2$$

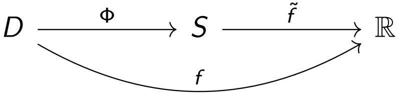
- This defines a functionn $\tilde{f} = f \circ \Phi^{-1} : S \to \mathbb{R}$, where

$$\tilde{f}(p) = f(x^1(p), x^2(p))$$

- Conversely, given a function $\tilde{f} : S \to \mathbb{R}$, we can define a function $f = \tilde{f} \circ \Phi : D \to \mathbb{R}$, where

$$f(x^1, x^2) = \tilde{f}(\Phi(x^1, x^2))$$

- We call f the pullback of $\tilde{f}$ by the map Φ



Differential of Function Using Coordinates

- Given $p \in S$ and $v = v^1\partial_1 + v^2\partial_2 \in T_pS$, the exterior derivative of $\tilde{f}$ is

$$\begin{aligned}
\langle v, d\tilde{f}(p)\rangle &= \left.\frac{d}{dt}\right|_{t=0} \tilde{f}(c(t)) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(x^1(t), x^2(t)) \\
&= \dot{x}^1\partial_1 f + \dot{x}^2\partial_2 f \\
&= v^1\partial_1 f + v^2\partial_2 f \\
&= \langle v^1\partial_1 + v^2\partial_2, dx^1\,\partial_1 f + dx^2\,\partial_2 f\rangle \\
&= \langle v, dx^1\,\partial_1 f + dx^2\,\partial_2 f\rangle
\end{aligned}$$

- Since this holds for any $v \in T_pS$, we get

$$d\tilde{f} = \partial_1 f\,dx^1 + \partial_2 f\,dx^2$$

Exterior Derivative of a Function

- Therefore,
$$d\tilde{f} = dx^1 \partial_1 f + dx^2 \partial_2 f$$
- It follows that, given a function $f : D \to \mathbb{R}$,
- For convenience, we will also write $\tilde{f}$ as just f
- So when we write $f(x^1, x^2)$, we sometimes really mean $f(x^1(p), x^2(p))$

Confusing notation

- Depending on the context, there two possible meanings of $f(x^1, x^2)$
- If there is no abstract surface anywhere, then $f(x^1, x^2)$ is just a function on a domain in $\mathbb{R}^2$
- If there is an abstract surface and a coordinate map $\Phi(x^1, x^2)$, then $f(x^1, x^2)$ is the function $\tilde{f}(p) = f(x^1(p), x^2(p))$
- Either way,

$$df = \partial_1 f \, dx^1 + \partial_2 f \, dx^2$$

- If we write f as $f(s, t)$, then

$$df = \partial_s f \, ds + \partial_t f \, dt$$

Examples

- 1-forms

$$\alpha = x\,dx + y\,dy$$
$$\theta = \frac{-y\,dx + x\,dy}{x^2 + y^2} = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy$$

- Exterior derivative of a function

$$d(xy) = \partial_x(xy)\,dx + \partial_y(xy)\,dy = y\,dx + x\,dy$$
$$d(u^2+v^2) = \partial_u(u^2+v^2)\,du + \partial_v(u^2+v^2)\,dv = 2u\,du + 2v\,dv$$

Line Integral of a Vector Field along a Curve

- Recall that in Calculus III a line integral was an integral of a vector field V along a parameterized curve $c : [a, b] \to \mathbb{R}^m$:

$$\int_c V \cdot d\vec{r} = \int_{t=a}^{t=b} V(c(t)) \cdot c'(t)\, dt$$

- This requires the dot product, which we want to avoid
- Observation: For each t, the integrand is a linear function of $c'(t) \in \widehat{\mathbb{R}}^m$
- Therefore, the integrand is the value of a 1-form evaluated on the vector $c'(t)$
- Conclusion: The natural dot-product-free thing to integrate is a differential 1-form

Abstract Definition of a Line Integral

- Let $C \subset S$ be an oriented curve in S with a parameterization

$$c : [t_{\text{start}}, t_{\text{end}}] \to S$$

  - We do not assume that $t_{\text{start}} \leq t_{\text{end}}$
- Given a 1-form θ on S, we write the line integral of θ on C to be

$$\int_C \theta$$

- The abstract definition of the line integral is

$$\int_c \theta = \int_{t=a}^{t=b} \langle \theta(c(t)), c'(t) \rangle \, dt$$

- This shows that the value of the line integral does not depend on the coordinates on S
- The value of the integral also does not depend on the parameterization of C