MATH-UA 377 Differential Geometry
Dual Vector Space
Tensors
Symmetric and Antisymmetric 2-Tensors
Tensor, Symmetric, and Wedge Products

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LIVE TRANSCRIPTION

Dual Vector Space

- Let $\mathbb{V}$ be an m-dimensional vector space
- The dual vector space of $\mathbb{V}$ is the vector space

$$\mathbb{V}^* = \{ \text{ linear functions } \mathbb{V} \to \mathbb{R}\}$$

- Given any $v \in \mathbb{V}$ and $\ell \in \mathbb{V}^*$, denote

$$\langle \ell, v \rangle = \langle v, \ell \rangle = \ell(v)$$

Dual Basis

- Given a basis of $\mathbb{V}$,

$$B = \begin{bmatrix} b_1 & \dots & b_m \end{bmatrix}$$

the dual basis is the basis of $\mathbb{V}^*$,

$$B^* = \begin{bmatrix} \beta^1 \\ \vdots \\ \beta^m \end{bmatrix},$$

where

$$\langle \beta^k, b_j \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

- Equivalently, for any $v = a^1 b_1 + \cdots a^m b_m \in \mathbb{V}$,

$$\langle \beta^k, v \rangle = \langle \beta^k, a^1 b_1 + \cdots a^m b_m \rangle = a^k$$

Evaluation Using Basis and Dual Basis

If

$$v = v^k b_k \text{ and } \theta = \theta_k \beta^k,$$

then

$$\begin{aligned}
\langle v, \theta \rangle &= \langle v^j b_j, \theta_k \beta^k \rangle \\
&= v^j \theta_k \langle b_j, \beta^k \rangle \\
&= v^k \theta_k
\end{aligned}$$

Tensors

- A tensor on a vector space $\mathbb{V}$ is a multilinear function on $\mathbb{V}$.
- A 1-tensor is a linear function $\ell : \mathbb{V} \to \mathbb{R}$ and therefore a covector $\ell \in \mathbb{V}^*$.
- A 2-tensor is a bilinear function

$$\begin{aligned} \tau : \mathbb{V} \times \mathbb{V} &\to \mathbb{R} \\ (v_1, v_2) &\mapsto \tau(v_1, v_2). \end{aligned}$$

- Bilinear means linear with respect to each input:

$$\begin{aligned} \tau(v_1 + w_1, v_2) &= \tau(v_1, v_2) + \tau(w_1, v_2) \\ \tau(c_1 v_1, v_2) &= c_1 \tau(v_1, v_2) \\ \tau(v_1, v_2 + w_2) &= \tau(v_1, v_2) + \tau(v_1, w_2) \\ \tau(v_1, c_2 v_2) &= c_2 \tau(v_1, v_2). \end{aligned}$$

- The space of 2-tensors on $\mathbb{V}$ will be denoted

$$\mathbb{V}^* \otimes \mathbb{V}^*$$

Examples of 2-Tensors

- The dot product on a Euclidean vector space,

$$\tau(v, w) = v \cdot w$$

  is a 2-tensor
- Given a square matrix

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mm} \end{bmatrix},$$

  the function $\tau : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\begin{aligned} \tau(a, b) &= a^T M b \\ &= \begin{bmatrix} a^1 & \cdots & a^m \end{bmatrix} \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mm} \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^m \end{bmatrix} \\ &= \sum_{j,k=1}^{m} a^j b^k M_{jk} = a^j b^k M_{jk} \text{ is a 2-tensor} \end{aligned}$$

2-Tensor as a Matrix

- Consider a 2-tensor τ on an m-dimensional vector space $\mathbb{V}$
- Given a basis $(e_1, \ldots, e_m)$ of $\mathbb{V}$, let

$$M_{ij} = \tau(e_i, e_j)$$

- If

$$v = a^1 e_1 + \cdots + a^m e_m \text{ and } w = b^1 e_1 + \cdots + b^m e_m,$$

  then

$$\begin{aligned}
\tau(v, w) &= \tau(a^1 e_1 + \cdots + a^m e_m, b^1 e_1 + \cdots + b^m e_m) \\
&= a^i b^j \tau(e_i, e_j) \\
&= \begin{bmatrix} a^1 & \cdots & a^m \end{bmatrix} \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mm} \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^m \end{bmatrix} = a^T M b
\end{aligned}$$

Tensor product

- Given $\ell_1, \ell_2 \in \mathbb{V}^*$, define their tensor product to be the 2-tensor, which we denote by $\ell_1 \otimes \ell_2$, to be

$$(\ell_1 \otimes \ell_2)(v_1, v_2) = \ell_1(v_1)\ell_2(v_2)$$

- Note that if $\ell_1 \neq \ell_2$, then

$$\ell_1 \otimes \ell_2 \neq \ell_2 \otimes \ell_1$$

Basis

- If $v = v^k b_k$ and $w = w^k b_k$, then
$$(\beta^j \otimes \beta^k)(v, w) = \beta^j(v)\beta^k(w) = v^j w^k$$
- If τ is a 2-tensor, then
$$\begin{aligned}\tau(v, w) &= \tau(v^1 b_1 + \cdots + v^m b_m, w^1 b_1 + \cdots w^m b_m)\\ &= v^j w^k \tau(b_j, b_k)\\ &= M_{jk}(\beta^j \otimes \beta^k)(v, w)\\ &= (M_{jk}\beta^j \otimes \beta^k)(v, w)\end{aligned}$$
- Since this holds for any v, w, it follows that, as functions of v and w,
$$\tau = M_{jk}\beta^j \otimes \beta^k$$
- If $M_{jk}\beta^j \otimes \beta^k = 0$, then
$$0 = (M_{jk}\beta^j \otimes \beta^k)(b_j, b_k) = M_{jk}$$
and therefore $\beta^j \otimes \beta^k$, $1 \leq j, k \leq m$, are linearly independent
- $\{\beta^j \otimes \beta^k,\ 1 \leq j, k \leq m\}$ is a basis of $\mathbb{V}^* \otimes \mathbb{V}^*$

Dimension of $\mathbb{V}^* \otimes \mathbb{V}^*$

- If
$$(b_1, \dots, b_m)$$
is a basis of $\mathbb{V}$
- And $(\beta^1, \dots, \beta^m)$ is the dual basis of $\mathbb{V}^*$
- Then $\beta^j \otimes \beta^k$, $1 \leq j, k \leq m$, is a basis of $\mathbb{V}^* \otimes \mathbb{V}^*$
- Therefore
$$\dim \mathbb{V}^* \otimes \mathbb{V}^* = (\dim V)^2$$

Symmetric 2-tensors

- A 2-tensor τ is *symmetric*, if

$$\tau(w, v) = \tau(v, w), \ \forall \ v, w \in \mathbb{V}.$$

- A 2-tensor τ is symmetric if and only if the matrix M is symmetric, i.e., $M^T = M$
- The space of all symmetric 2-tensors is a linear subspace of the vector space of 2-tensors
- The dot product is an example of a symmetric 2-tensor

Symmetric product

- Given $\ell_1, \ell_2 \in \mathbb{V}^*$, denote

$$\ell_1 \circ \ell_2 = \frac{1}{2}(\ell_1 \otimes \ell_2 + \ell_2 \otimes \ell_1)$$

- $\ell_1 \circ \ell_2 = \ell_2 \circ \ell_1$
- If $v, w \in \mathbb{V}$, then

$$\begin{aligned}(\ell_1 \otimes \ell_2)(v, w) &= \frac{1}{2}(\ell_1(v)\ell_2(w) + \ell_2(v)\ell_1(w)) \\ &= (\ell_1 \otimes \ell_2)(w, v)\end{aligned}$$

- Therefore, $\ell_1 \circ \ell_2 \in S^2\mathbb{V}^*$
- If $(\beta^1, \dots, \beta^m)$ is a basis of $S^2\mathbb{V}^*$, then

$$\beta_j \circ \beta_k,\ 1 \le j \le k \le m$$

  is a basis of $S^2\mathbb{V}^*$
- Therefore, if $m = \dim \mathbb{V}$, then

$$\dim S^2\mathbb{V}^* = \frac{1}{2}m(m+1)$$

Antisymmetric or exterior 2-tensors

- A 2-tensor τ is *antisymmetric* or *exterior*, if

$$\tau(w, v) = -\tau(v, w), \ \forall \ v, w \in \mathbb{V}.$$

- If τ is antisymmetric, then $\tau(v, v) = 0$.
- The space of all antisymmetric 2-tensors is a linear subspace of the space of 2-tensors
- An example of an antisymmetric 2-tensor on $\mathbb{R}^2$ is

$$\delta(\langle a^1, a^2\rangle, \langle b^1, b^2\rangle) = \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix} = a^1 b^2 - a^2 b^1.$$

Exterior 2-Tensor on 2-Dimensional Vector Space

- Let $\mathbb{V}$ be 2-dimensional
- Let τ be an exterior 2-tensor on $\mathbb{V}$
- Let (e_1, e_2) be a basis of $\mathbb{V}$
- $c = \tau(e_1, e_2)$
- Given any vectors $v = a^1 e_1 + a^2 e_2$ and $w = b^1 e_1 + b^2 e_2$,

$$\begin{aligned}
\tau(v, w) &= \tau(a^1 e_1 + a^2 e_2, b^1 e_1 + b^2 e_2) \\
&= a^1 b^1 \tau(e_1, e_1) + a^1 b_2 \tau(e_1, e_2) + a^2 b_1 \tau(e_2, e_1) + a^2 b^2 \tau(e_2, \\
&= \tau(e_1, e_2)(a^1 b_2 - a^2 b_1) \\
&= c \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix}
\end{aligned}$$

- The space of exterior 2-tensors on a 2-dimensional vector space is a 1-dimensional vector space
- The space of all exterior 2-tensors will be denoted $\Lambda^2 \mathbb{V}^*$

Notation

- If $\tau \in \mathbb{V}^* \otimes \mathbb{V}^*$ and $v, w \in \mathbb{V}$, then we will write
$$\tau(v, w) = \langle v \otimes w, \tau \rangle = \langle \tau, v \otimes w \rangle$$
- If $\tau \in S^2\mathbb{V}^*$, then
$$\langle v \otimes w, \tau \rangle = \langle w \otimes v, \tau \rangle$$
- If $\tau \in \Lambda^2\mathbb{V}^*$, then
$$\langle v \otimes w, \tau \rangle = -\langle w \otimes v, \tau \rangle$$

Wedge product of two covectors

- The exterior or wedge product of $\ell^1, \ell^2 \in \mathbb{V}^*$ is the exterior 2-tensor
$$\theta^1 \wedge \theta^2 = \theta^1 \otimes \theta^2 - \theta^2 \otimes \theta^1.$$
- $\theta^1 \wedge \theta^2 = -\theta^2 \wedge \theta^1$
- $\theta \wedge \theta = 0$
- In particular,
$$\begin{aligned}\langle v \otimes w, \theta^1 \wedge \theta^2\rangle &= \langle v \otimes w, \theta^1 \otimes \theta^2 - \theta^2 \otimes \theta^1\rangle \\ &= \langle v, \theta^1\rangle\langle w, \theta^2\rangle - \langle w, \theta^1\rangle\langle v, \theta^2\rangle \\ &= -\langle w \otimes v, \theta^1 \wedge \theta^2\rangle\end{aligned}$$
- Therefore, $\theta^1 \wedge \theta^2 \in \Lambda^2\mathbb{V}^*$

Wedge products of dual basis covectors

- Let $E = (e_1, \dots, e_m)$ be a basis of $\mathbb{V}$ and $\Theta = (\theta^1, \dots, \theta^m)$ the dual basis
- If $1 \le i, j, k, l \le m$, then

$$\begin{aligned} \langle e_i \otimes e_j, \theta^k \wedge \theta^l \rangle &= \langle e_i \otimes e_j, \theta^k \otimes \theta^l - \theta^l \otimes \theta^k \rangle \quad = \langle e_i, \theta^k \rangle \langle e_j, \theta^l - \langle e \\ &= \delta_i^k \delta_j^l - \delta_i^l \delta_j^k \end{aligned}$$

- If $i = j$ or $k = l$, then

$$\langle e_i \otimes e_j, \theta^k \wedge \theta^l \rangle = 0$$

- If $i \neq j$ and $k \neq l$, then

$$\langle e_i \otimes e_j, \theta^k \wedge \theta^l \rangle = \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ -1 & \text{if } i = l \text{ and } j = k \\ 0 & \text{otherwise} \end{cases}$$

Basis of $\Lambda^2\mathbb{V}^*$

- Suppose $\tau \in \Lambda^2\mathbb{V}$ and

$$M_{ij} = \langle \tau, e_i \otimes e_j \rangle = -M_{ji}$$

- Given $v = v^i e_i, w = w^j e_j \in \mathbb{V}$,

$$\begin{aligned} \langle \tau, v \otimes w \rangle &= \langle \tau, (v^i e_i) \otimes (w^j e_j) \rangle \\ &= v^i w^j \langle \tau, e_i \otimes e_j \rangle = M_{ij} v^i w^j \end{aligned}$$

- On the other hand,

$$\begin{aligned} \langle M_{kl}\theta^k \wedge \theta^l, v \otimes w \rangle &= M_{kl} \langle \theta^k \wedge \theta^l, (v^i e_i) \otimes (w^j e_j) \rangle \\ &= M_{kl} v^k w^l \langle \theta^k \wedge \theta^l, e_i \otimes e_j \rangle \\ &= M_{kl} v^k w^l (\delta^k_i \delta^l_j - \delta^l_i \delta^k_j) \\ &= M_{kl} (v^k b^w - v^l w^k) \\ &= 2 M_{ij} v^i w^j \end{aligned}$$

- Therefore, $\tau = \dfrac{1}{2} M_{ij} \theta^i \wedge \theta^j$

Dimension of $\Lambda^2 \mathbb{V}^*$

- If $(e_1, \dots, e_m)$ is a basis of $\mathbb{V}$ and $(\theta^1, \dots, \theta^m)$ is the dual basis, then
$$\theta^j \wedge \theta^k, 1 \leq j < k \leq m$$
comprise a basis of $\Lambda^2 \mathbb{V}^*$
- Therefore,
$$\dim \Lambda^2 \mathbb{V}^* = \binom{m}{2} = \frac{1}{2} m(m-1)$$