MATH-UA 377 Differential Geometry
Dual Vector Space
Tensors

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LIVE TRANSCRIPTION

Dual Vector Space

- Let $\mathbb{V}$ be an m-dimensional vector space
- The dual vector space of $\mathbb{V}$ is the vector space

$$\mathbb{V}^* = \{ \text{ linear functions } \mathbb{V} \to \mathbb{R}\}$$

- There is a natural function

$$\begin{aligned}\mathbb{V} \times \mathbb{V}^* &\to \mathbb{R} \\ (v, \ell) &\mapsto \langle \ell, v\rangle = \langle v, \ell\rangle = \ell(v)\end{aligned}$$

- There is a natural linear map

$$\begin{aligned}\mathbb{V} &\to (\mathbb{V}^*)^* \\ v &\mapsto f_v,\end{aligned}$$

  where

$$f_v(\ell) = \ell(v)$$

  - Easy to check this is an isomorphism

Dual Basis

- Given a basis of $\mathbb{V}$,

$$B = \begin{bmatrix} b_1 & \dots & b_m \end{bmatrix}$$

  the dual basis is the basis

$$B^* = \begin{bmatrix} \beta^1 \\ \vdots \\ \beta^m \end{bmatrix},$$

  where

$$\langle \beta^k, b_j \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

- Equivalently, for any $v = a^1 b_1 + \cdots a^m b_m \in \mathbb{V}$,

$$\langle \beta^k, v \rangle = \langle \beta^k, a^1 b_1 + \cdots a^m b_m \rangle = a^k$$

The Dual Basis is a Basis of the Dual Vector Space

- If $\ell \in \mathbb{V}^*$, let
$$c_k = \langle \ell, b_k \rangle$$
- Recall that if $v = a^k b_k$, then
$$\langle \beta^k, v \rangle = a^k$$
- Therefore, given $\ell \in \mathbb{V}^*$,
$$\begin{aligned} \langle \ell, v \rangle &= \langle \ell, a^k b_k \rangle \\ &= a^k \langle \ell, b_k \rangle \\ &= \langle \beta^k, v \rangle c_k \\ &= \langle c_k \beta^k, v \rangle \end{aligned}$$
- Therefore, ℓ and $c_k \beta^k$ define the same function on $\mathbb{V}$, i.e.,
$$\ell = c_k \beta^k$$
- Moreover, $\ell = 0$ if and only if $c_1 = \cdots = c_m = 0$
- Therefore, $(\beta^1, \ldots, \beta^m)$ is a basis of $\mathbb{V}^*$

Evaluation Using Basis and Dual Basis

If

$$v = v^k b_k \text{ and } \theta = \theta_k \beta^k,$$

then

$$\begin{aligned} \langle v, \theta \rangle &= \langle v^j b_j, \theta_k \beta^k \rangle \\ &= v^j \theta_k \langle b_j, \beta^k \rangle \\ &= v^k \theta_k \end{aligned}$$

Tensors

- A tensor on a vector space $\mathbb{V}$ is a multilinear function on $\mathbb{V}$.
- A 1-tensor is a linear function $\ell : \mathbb{V} \to \mathbb{R}$ and therefore a covector.
- The set of all 1-tensors is $\mathbb{V}^*$.
- A 2-tensor is a bilinear function

$$\begin{aligned} \tau : \mathbb{V} \times \mathbb{V} &\to \mathbb{R} \\ (v_1, v_2) &\mapsto \tau(v_1, v_2). \end{aligned}$$

- Bilinear means linear with respect to each input:

$$\begin{aligned} \tau(v_1 + w_1, v_2) &= \tau(v_1, v_2) + \tau(w_1, v_2) \\ \tau(c_1 v_1, v_2) &= c_1 \tau(v_1, v_2) \\ \tau(v_1, v_2 + w_2) &= \tau(v_1, v_2) + \tau(v_1, w_2) \\ \tau(v_1, c_2 v_2) &= c_2 \tau(v_1, v_2). \end{aligned}$$

- The space of 2-tensors on $\mathbb{V}$ will be denoted

$$\mathbb{V}^* \otimes \mathbb{V}^*$$

Examples of 2-Tensors

- The dot product on a Euclidean vector space,
$$\tau(v,w) = v \cdot w$$
is a 2-tensor
- Given a square matrix
$$M = \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mm} \end{bmatrix},$$
the function $\tau : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ defined by
$$\begin{aligned} \tau(a,b) &= a^T M b \\ &= \begin{bmatrix} a^1 & \cdots & a^m \end{bmatrix} \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mm} \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^m \end{bmatrix} \\ &= \sum_{j,k=1}^{m} a^j b^k M_{jk} = a^j b^k M_{jk} \text{ is a 2-tensor} \end{aligned}$$

2-Tensor as a Matrix

- Consider a 2-tensor τ on an m-dimensional vector space $\mathbb{V}$
- Given a basis $(e_1, \ldots, e_m)$ of $\mathbb{V}$, let

$$M_{ij} = \tau(e_i, e_j)$$

- If

$$v = a^1 e_1 + \cdots + a^m e_m \text{ and } w = b^1 e_1 + \cdots + b^m e_m,$$

then

$$\begin{aligned}
\tau(v, w) &= \tau(a^1 e_1 + \cdots + a^m e_m, b^1 e_1 + \cdots + b^m e_m) \\
&= a^i b^j \tau(e_i, e_j) \\
&= \begin{bmatrix} a^1 & \cdots & a^m \end{bmatrix} \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & & \vdots \\ M_{m1} & \cdots & M_{mm} \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^m \end{bmatrix} = a^T M b
\end{aligned}$$

Tensor product

- Given $\ell_1, \ell_2 \in \mathbb{V}^*$, define their tensor product to be the 2-tensor, which we denote by $\ell_1 \otimes \ell_2$, to be

$$(\ell_1 \otimes \ell_2)(v_1, v_2) = \ell_1(v_1)\ell_2(v_2)$$

- If $v = v^k b_k$ and $w = w^k b_k$, then

$$(\beta^j \otimes \beta^k)(v, w) = \beta^j(v)\beta^k(w) = v^j w^k$$

Basis and Dimension of $\mathbb{V}^* \otimes \mathbb{V}^*$

- Given $\tau \in \mathbb{V}^* \otimes \mathbb{V}^*$, let

$$M_{jk} = \tau(b_j, b_k),$$

- Then for any $v, w \in \mathbb{V}$,

$$\begin{aligned}
\tau(v, w) &= \tau(v^j b_j, w^k b_k) \\
&= v^j w^k \tau(b_j, b_k) \\
&= (\beta^j \otimes \beta^k)(v, 2) M_{jk} \\
&= (M_{jk} \beta^j \otimes \beta^k)(v, w)
\end{aligned}$$

- Therefore, as functions,

$$\tau = M_{jk} \beta^j \otimes \beta^k,$$

  and $\tau = 0$ if and only if $M_{jk} = 0$ for all $1 \leq j, k \leq m$
- It follows that $\beta^j \otimes \beta^k$, $1 \leq j, k \leq m$ form a basis of $\mathbb{V}^* \otimes \mathbb{V}^*$
- Therefore,

$$\dim \mathbb{V}^* \otimes \mathbb{V}^* = (\dim \mathbb{V})^2$$

Symmetric 2-tensors

- A 2-tensor τ is *symmetric*, if

$$\tau(w, v) = \tau(v, w), \ \forall \ v, w \in \mathbb{V}.$$

- A 2-tensor τ is symmetric if and only if the matrix M is symmetric, i.e., $M^T = M$
- The space of all symmetric 2-tensors is a linear subspace of the vector space of 2-tensors
- The dot product is an example of a symmetric 2-tensor.

Exterior 2-tensors

- A 2-tensor τ is *antisymmetric* or *exterior*, if

$$\tau(w, v) = -\tau(v, w), \ \forall \ v, w \in \mathbb{V}.$$

- If τ is antisymmetric, then $\tau(v, v) = 0$.
- A 2-tensor τ is antisymmetric if and only if the matrix M is antisymmetric, i.e., $M^T = -M$
- The space of all antisymmetric 2-tensors is a linear subspace of the space of 2-tensors
- An example of an antisymmetric 2-tensor on $\mathbb{R}^2$ is

$$\delta(\langle a^1, a^2\rangle, \langle b^1, b^2\rangle) = \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix} = a^1 b^2 - a^2 b^1.$$

Exterior 2-Tensor on 2-Dimensional Vector Space

- Let $\mathbb{V}$ be 2-dimensional
- Let τ be an exterior 2-tensor on $\mathbb{V}$
- Let (e_1, e_2) be a basis of $\mathbb{V}$
- $c = \tau(e_1, e_2)$
- Given any vectors $v = a^1e_1 + a^2e_2$ and $w = b^1e_1 + b^2e_2$,

$$\begin{aligned}\tau(v, w) &= \tau(a^1e_1 + a^2e_2, b^1e_1 + b^2e_2) \\ &= a^1b^1\tau(e_1, e_1) + a^1b_2\tau(e_1, e_2) + a^2b_1\tau(e_2, e_1) + a^2b^2\tau(e_2, \\ &= \tau(e_1, e_2)(a^1b_2 - a^2b_1) \\ &= c\det\begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix}\end{aligned}$$

- The space of exterior 2-tensors on a 2-dimensional vector space is a 1-dimensional vector space

Notation

- The space of 2-tensors on a vector space $\mathbb{V}$ is $\mathbb{V}^* \otimes \mathbb{V}^*$.
- The space of symmetric 2-tensors on $\mathbb{V}$ is $S^2\mathbb{V}^*$.
- The space of exterior 2-tensors on $\mathbb{V}$ is $\bigwedge^2 \mathbb{V}^*$.
- Given a 2-tensor τ and vectors $v, w \in \mathbb{V}$, we will write

$$\tau(v, w) = \langle \tau, v \otimes w \rangle = \langle v \otimes w, \tau \rangle$$

- The symbol $\otimes$ has the following rules:

$$\begin{aligned}(a^1 v_1 + a^2 v_2) \otimes w &= a^1(v_1 \otimes w) + a^2(v_2 \otimes w) \\ v \otimes (b^1 w_1 + b^2 w_2) &= b^1(v \otimes w_1) + b^2(v \otimes w_2)\end{aligned}$$

- IMPORTANT: $v \otimes w \neq w \otimes v$ unless v is a scalar product of w or vice versa

Tensor product

- Given $\theta^1, \theta^2 \in \mathbb{V}^*$, their tensor product is the 2-tensor $\theta^1 \otimes \theta^2$ defined by

$$\langle v \otimes w, \theta^1 \otimes \theta^2 \rangle = \langle v, \theta^1 \rangle \langle w, \theta^2 \rangle.$$

- Note that $\theta^2 \otimes \theta^1 \neq \theta^1 \otimes \theta^2$ unless one is a scalar product of the other
- The symmetric product of θ^1 and θ^2 is the symmetric 2-tensor

$$\theta^1 \circ \theta^2 = \theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1.$$

  In particular, given any $v, w \in \mathbb{V}$,

$$\langle v \otimes w, \theta^1 \circ \theta^2 \rangle = \langle v, \theta^1 \rangle \langle w, \theta^2 \rangle + \langle w, \theta^1 \rangle \langle v, \theta^2 \rangle$$

- Note that $\theta^2 \circ \theta^1 = \theta^1 \circ \theta^2$.

Wedge product of two covectors

- The exterior or wedge product of θ^1 and θ^2 is the exterior 2-tensor
$$\theta^1 \wedge \theta^2 = \theta^1 \otimes \theta^2 - \theta^2 \otimes \theta^1.$$
- In particular,
$$\begin{aligned}\langle v \otimes w, \theta^1 \wedge \theta^2 \rangle &= \langle v \otimes w, \theta^1 \otimes \theta^2 - \theta^2 \otimes \theta^1 \rangle \\ &= \langle v, \theta^1 \rangle \langle w, \theta^2 \rangle - \langle w, \theta^1 \rangle \langle v, \theta^2 \rangle\end{aligned}$$
- $\theta^2 \wedge \theta^1 = -\theta^1 \wedge \theta^2$.
- $\theta \wedge \theta = 0$.

Exterior 2-tensors With Respect To Basis

- Let $E = (e_1, \ldots, e_m)$ be a basis of $\mathbb{V}$ and $\Theta = (\theta^1, \ldots, \theta^m)$ the dual basis
- The exterior 2-form $\theta^i \wedge \theta^j$ has the following properties:

$$\begin{aligned}
\langle \theta^i \wedge \theta^j, e_k \otimes e_l \rangle &= \langle \theta^i \otimes \theta^j - \theta^j \otimes \theta^i, e_k \otimes e_l \rangle \\
&= \langle \theta^i, e_k \rangle \langle \theta^j, e_l \rangle - \langle \theta^j, e_k \rangle \langle \theta^i, e_l \rangle \\
&= \begin{cases} 1 & \text{if } i = k \text{ and } j = l \\ 1 & \text{if } i = l \text{ end } j = k \\ 0 & \text{otherwise} \end{cases} \\
&= \delta^i_k \delta^j_l - \delta^j_k \delta^i_l
\end{aligned}$$

Dimension of $\bigwedge^2 \mathbb{V}$

- Suppose $\tau \in \bigwedge^2 \mathbb{V}$ and
$$M_{ij} = \langle \tau, e_i \otimes e_j \rangle = -M_{ji}$$
- Given $v = a^i e_i, w = b^i e_i \in \mathbb{V}$,
$$\begin{aligned}\langle \tau, v \otimes w \rangle &= \langle \tau, (a^i e_i) \otimes (b^j e_j) \rangle \\ &= a^i b^j \langle \tau, e_i \otimes e_j \rangle = M_{ij} a^i b^j\end{aligned}$$
- On the other hand,
$$\begin{aligned}\langle M_{ij} \theta^i \wedge \theta^j, v \otimes w \rangle &= M_{ij} \langle \theta^i \wedge \theta^j, (a^k e_k) \otimes (b^l e_l) \rangle \\ &= M_{ij} a^k b^l \langle \theta^i \wedge \theta^j, e_k \otimes e_l \rangle \\ &= M_{ij} (a^i b^j - a^j b^i) = 2 M_{ij} a^i b^j\end{aligned}$$
- Therefore, τ is uniquely determined by the matrix M,
$$\tau = \frac{1}{2} M_{ij} \theta^i \wedge \theta^j$$
- It follows that $\dim \bigwedge^2 V^* = \binom{m}{2} = \frac{n(n-1)}{2}$