MATH-UA 377 Differential Geometry
Directional Derivative of a Function
Differential of a Function

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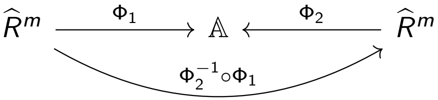
LIVE TRANSCRIPTION

Affine Coordinate Chart of Affine Space

- Given $p \in O$ and a basis $(e_1, \ldots, e_m)$ of $\mathbb{V}$, let

$$\begin{aligned} \Phi : \widehat{R}^m &\to \mathbb{A} \\ \langle a^1, \ldots, a^m \rangle &\mapsto p + e_1 a^1 + \cdots + e_m a^m \end{aligned}$$

- Φ is a parameterization of $\mathbb{A}$ by $\widehat{R}^m$
- Φ is a coordinate chart for $\mathbb{A}$
- Φ is an affine map
- Given any two affine coordinate charts Φ_1 and Φ_2,



the map $\Phi_2^{-1} \circ \Phi_1 : \widehat{R}^m \to \widehat{R}^m$ is an affine and therefore C^∞ isomorphism of $\widehat{R}^m$

Open Subsets of Affine Space

- A set $O \subset \mathbb{A}$ is defined to be **open** if $\Phi^{-1}(O) \subset \widehat{R}^m$ is open

$$\begin{array}{ccc} \widehat{R}^m & \xrightarrow{\quad\Phi\quad} & \mathbb{A} \\ \cup & & \cup \\ \Phi^{-1}(O) & \xrightarrow{\quad\Phi\quad} & O \end{array}$$

- This definition does not depend on the affine coordinate chart used

$$\begin{array}{ccc} \Phi_1^{-1}(O) & \xrightarrow{\quad\Phi_1\quad} & O \\ {\scriptstyle \Phi_2^{-1}\circ\Phi_1}\Big\downarrow & \nearrow{\scriptstyle \Phi_2} & \\ \Phi_2^{-1}(O) & & \end{array}$$

C^k Functions on Affine Space

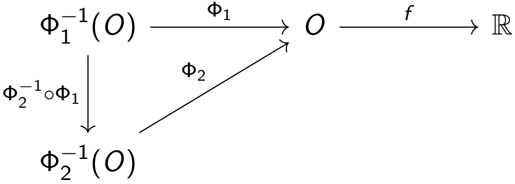
- A function $f : O \to \mathbb{R}$ is C^k if the function

$$f \circ \Phi : \Phi^{-1}(O) \to \mathbb{R}$$

  is C^k

$$\begin{array}{ccccc} \widehat{R}^m & \xrightarrow{\Phi} & \mathbb{A} & & \\ \cup & & \cup & & \\ \Phi^{-1}(O) & \xrightarrow{\Phi} & O & \xrightarrow{f} & \mathbb{R} \end{array}$$

- This definition does not depend on the affine coordinate charts used



Directional Derivative

- Consider a C^1 function $f : \mathbb{A}^m \to \mathbb{R}$
- Given a curve $c : I \to \mathbb{A}^m$,

$$f \circ c : I \to \mathbb{R},$$

  which can be differentiated
- Given $v \in \mathbb{V}^m$, define the directional derivative of f in the direction v at $p \in \mathbb{A}^3$ to be

$$D_v f(p) = \left.\frac{d}{dt}\right|_{t=0} (f \circ c)(t),$$

  where c is a curve such that $c(0) = p$ and $\dot{c}(0) = v$

Parameterized Curve With Respect to Coordinates

- Since $\Phi(a^1, \ldots, a^m) = p + e_k a^k$,

$$(f \circ \Phi)(a^1, \ldots, a^m)) = f(p + e_k a^k)$$

- Given a curve c, we can write

$$c(t) = p + e_k a^k(t),$$

- If $c(0) = p$ and $\dot{c} = v = v^k e_k$, then

$$\langle a^1(0), \ldots, a^m(0) \rangle = 0$$
$$\langle \dot{a}^1(0), \ldots, \dot{a}^m \rangle = \langle v^1, \ldots, v^m \rangle$$

Directional Derivative With Respect To Coordinates

- By the chain rule,

$$\begin{aligned} D_v f(p) &= \left.\frac{d}{dt}\right|_{t=0} f(c(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} (f \circ \Phi)(a^1(t), \dots, a^m(t)) \\ &= \partial_k (f \circ \Phi)(0)\dot{a}^k(0) \\ &= v^k \partial_k (f \circ \Phi)(0) \end{aligned}$$

- Therefore, $D_v f(p)$ is a linear function of v

Differential of a Function on Affine Space

- This means that for each $p \in O$, there is a linear function

$$\begin{aligned} df(p) : \mathbb{V} &\to \mathbb{R} \\ v &\mapsto D_v f(p) \end{aligned}$$

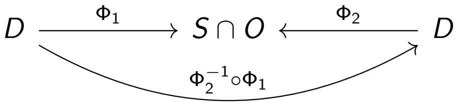
- Recall that a linear function on $\mathbb{V}$ is an element of the dual space $\mathbb{V}^*$
- Therefore, for each p, $df(p) \in V^*$, which is called the **differential of f at p**
- This in turn defines a map

$$\begin{aligned} df : O &\to \mathbb{V}^* \\ p &\mapsto df(p) \end{aligned}$$

which is called the **differential of f**

C^k Coordinate Charts on a Surface

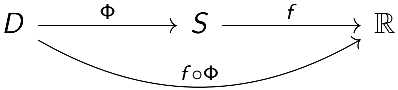
- A set $\Omega \subset S$ is defined to be **open** if there is an open subset $O \subset \mathbb{A}^3$ such that
$$\Omega = S \cap O$$
- The image $\Phi(D) = S \cap O$ of any coordinate chart is open
- A surface is C^k if each coordinate chart $\Phi : D \to S \cap O \subset \mathbb{A}$ is a C^k map
- This does not depend on the coordinate chart used
- If S is C^k and $\Phi_1(D) = \Phi_2(D)$, then $\Phi_2^{-1} \circ \Phi_1$ is C^k



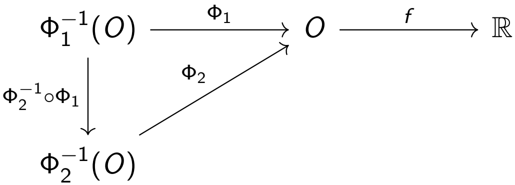
- Proved using implicit function theorem

C^k Functions on a Surface

- Consider a surface $S \subset \mathbb{A}^3$
- A function $f : S \to \mathbb{R}$ is C^k if for any coordinate chart $\Phi : D \to S \cap O$, the function $f \circ \Phi : D \to \mathbb{R}$ is C^k



- This definition does not depend on the coordinate charts used



Directional Derivative of Function on a Surface

- Defined exactly as for function on affine space
- Consider a C^1 function $f : S \to \mathbb{R}$, a point $p \in S$, and a tangent vector $v \in T_pS$
- Let $c : (-\delta, \delta) \to S$ be a C^1 curve such that

$$c(0) = p \text{ and } c'(0) = v$$

- The directional derivative of f in the direction v at p is

$$D_v f(p) = (f \circ c)'(0) = \left.\frac{d}{dt}\right|_{t=0} f(c(t))$$

Parameterized Curve on a Coordinate Chart

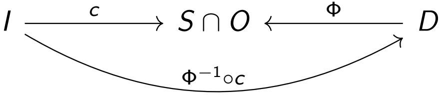
- Let $\Phi : D \to S$ be a coordinate chart such that $\Phi(0) = p \in S$
- The parameterization defines a basis (∂_1, ∂_2) of $T_p S$
- Given curve $c : I \to \Phi(D) \subset S \cap O$, there is the curve

$$\Phi^{-1} \circ c : I \to D \subset \mathbb{R}^2$$

  which can be written as

$$(\Phi^{-1} \circ c)(t) = (a^1(t), a^2(t)),$$

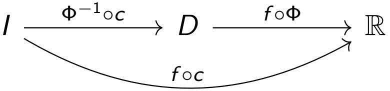
  where $c(t) = \Phi(a^1(t), a^2(t))$



Directional Derivative on a Coordinate Chart

- If $(\Phi^{-1} \circ c)(t) = (a^1(t), a^2(t))$, then
$$(\Phi^{-1} \circ c)'(t) = \langle \dot{a}^1(t), \dot{a}^2(t) \rangle$$
- If $\Phi(0) = p$ and $\dot{c}(0) = v^1\partial_1 + v^2\partial_2$, then
$$\langle \dot{a}^1(0), \dot{a}^2(0) \rangle = \langle v^1, v^2 \rangle$$
- By the chain rule, the directional derivative of f is

$$\begin{aligned} D_v(p) &= (f \circ c)'(0) \\ &= \frac{d}{dt}\bigg|_{t=0} (f \circ \Phi) \circ (\Phi^{-1} \circ c)(t) \\ &= \frac{d}{dt}\bigg|_{t=0} (f \circ \Phi)(a^1(t), a^2(t)) \\ &= v^1\partial_1(f \circ \Phi)(0,0) + v^2\partial_2(f \circ \Phi)(0,0) \end{aligned}$$



The Differential of a Function

- Again, $v \mapsto D_v f(p)$ is a linear function of $v \in T_p S$
- Therefore, it defines an element $df(p) \in T_p^* S$ such that

$$\langle v, df(p) \rangle = D_v f(p)$$

- Please note that the notation $\langle \cdot, \cdot \rangle$ used here is **different** from the same notation used for a vector $\langle v^1, v^2 \rangle \in \widehat{R}^2$
- Note that df is **NOT** a map from S to $\mathbb{V}^*$, because domain of

$$df(p) : T_p \to \mathbb{R}$$

  is $T_p S$, which is only a subset of $\mathbb{V}$
- df is an example of a **differential** 1-**form** on the surface S

Inverse Coordinate Map

- We can write a coordinate chart as $\Phi(x^1, x^2)$, where $(x^1, x^2) \in D$ is an input parameter
- Conversely, the inverse to Φ is

$$\begin{aligned} \Phi^{-1} : S \cap O &\to D \subset \mathbb{R}^2 \\ p &\mapsto (x^1(p), x^2(p)), \end{aligned}$$

  where x^1, x^2 are scalar functions on $S \cap O$
- We call x^1 and x^2 coordinate functions on $S \cap O$
- If $S = \mathbb{R}^2$, then (x^1, x^2) are the standard coordinates on $\mathbb{R}^2$

Directional Derivative of a Coordinate Function

- A curve c in $S \cap O$ can be written as

$$c(t) = \Phi(a^1(t), a^2(t)), \text{ where } (a^1(t), a^2(t)) = (\Phi^{-1} \circ c)(t))$$

- Equivalently,

$$x^k(c(t)) = a^k(t),\ \ k = 1, 2$$

- By the chain rule,

$$\dot{c} = \dot{a}^1 \partial_1 \Phi + \dot{a}^2 \partial_2 \Phi$$

- It follows that if $c(0) = p$ and $v = \dot{c}(0) = v^1 \partial_1 + v^2 \partial_2$, then

$$\begin{aligned} D_v x^1(p) &= \left.\frac{d}{dt}\right|_{t=0} x^1(c(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} a^1(t) \\ &= \dot{a}^1(0) \\ &= v^1 \end{aligned}$$

Differential of a Coordinate Function

- Given a coordinate chart $\Phi : D \to S \cap O$, $p \in \Phi(D)$, and $v = v^1\partial_1 + \in T_pS$, we have shown that

$$\begin{aligned} \langle v, dx^i \rangle &= D_v x^i(p) \\ &= v^i \end{aligned}$$

- In particular,

$$\langle \partial_j, dx^i \rangle = \delta^i_j$$

- This implies that (dx^1, dx^2) is the basis of T^*_pS dual to the basis of (∂_1, ∂_2)