MATH-UA 377 Differential Geometry
Parameterized Surface in $\mathbb{V}^3$
Coordinate Charts
Global Surface in $\mathbb{A}^3$
Tangent Space of a Surface

Deane Yang

Courant Institute of Mathematical Sciences
New York University

March 3, 2022

START RECORDING
LIVE TRANSCRIPTION

Embeddings

- Change of notation: Let $D \subset \mathbb{R}^2$ be an open set
- Consider a C^1 map $\Phi : D \to \mathbb{A}^3$
- Φ is nondegenerate if for every $(x^1, x^2) \in D$, the linear map

$$\partial\Phi(x^1, x^2) : \widehat{R}^2 \to \mathbb{V}^3$$

  has rank 2
- Φ is an **embedding** if it is nondegenerate and injective

Parameterized Surface in $\mathbb{A}^3$

- A parameterized surface consists of the following:
  - A set $S \subset O$
  - An embedding $\Phi : D \to \mathbb{A}^3$, such that

$$\Phi(D) = S$$

- Example: Paraboloid in $\mathbb{R}^3$

$$S = \{z = x^2 + y^2\}$$
$$\Phi(x, y) = (x, y, x^2 + y^2)$$

- Bad parameterization of paraboloid

$$\Phi(s, t) = (s^3, t^3, s^6 + t^6)$$

- Bad surface: Cone

$$S = \{z = \sqrt{x^2 + y^2}\}$$
$$\Phi(u, v) = (u, v, \sqrt{u^2 + v^2})$$

Global Surface in $\mathbb{A}^3$

- A sphere in $\mathbb{E}^3$ is *NOT* a parameterized surface
  - A circle is also not a parameterized curve
- Need a more general definition of a surface
- A sphere is a union of overlapping parameterized surfaces
- A parameterized surface is also called a **coordinate chart**
- Idea: A surface should be covered by a collection of overlapping coordinate charts

Precise Definition of a C^k Surface in $\mathbb{A}^3$

- Given a set $S \subset \mathbb{A}^3$, a coordinate chart on S consists of the following:
  - An open domain $D \subset \mathbb{R}^2$
  - An open subset $O \subset \mathbb{R}^3$
  - A C^k embedding $\Phi : Do \to O$

  such that

  $$\Phi(D) = S \cap O$$

- A set $S \subset \mathbb{A}^3$ is a C^k surface if for each $p \in S$, there is a coordinate chart containing p

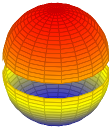
Sphere as Union of 6 Coordinate Charts

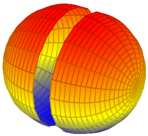
- $S = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ is a sphere of radius 1
- $D = \{(s,t) \in \mathbb{R}^2 \ : \ s^2 + t^2 < 1\}$ is a circular disk of radius 1
- Define open sets $O_k \subset \mathbb{R}^3$, $k = 1, 2, 3, 4, 5, 6$, where

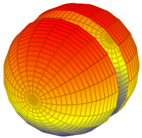
$$\begin{aligned}
O_1 &= \{(x,y,z) \ : \ z > 0\} \\
O_2 &= \{(x,y,z) \ : \ z < 0\} \\
O_3 &= \{(x,y,z) \ : \ y > 0\} \\
O_4 &= \{(x,y,z) \ : \ y < 0\} \\
O_5 &= \{(x,y,z) \ : \ x > 0\} \\
O_6 &= \{(x,y,z) \ : \ x < 0\}
\end{aligned}$$

- Each O_k is an open half-space lying on one side of a coordinate plane
- $S \cap O_k$ is an open hemisphere, which can be parameterized as a graph of the corresponding coordinate plane
- Since each point $(x,y,z) \in S$ has at least one nonzero coordinate, it lies in at least one of $O_1, O_2, O_3, O_4, O_5, O_6$

Sphere as Union of 6 Coordinate Charts







- Define coordinate maps $\Phi_k : D \to S \cap O_k,\ k = 1, 2, 3, 4, 5, 6,$ where

$$\Phi_1(s,t) = (s, t, \sqrt{1 - s^2 - t^2})$$

$$\Phi_2(s,t) = (s, t, -\sqrt{1 - s^2 - t^2})$$

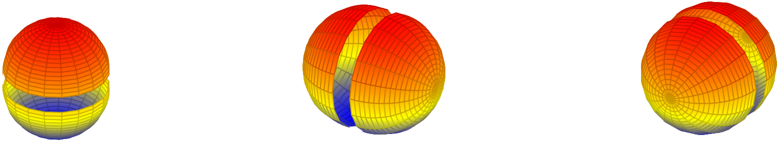
$$\Phi_3(s,t) = (s, \sqrt{1 - s^2 - t^2}, t)$$

$$\Phi_4(s,t) = (s, -\sqrt{1 - s^2 - t^2}, t)$$

$$\Phi_5(s,t) = (\sqrt{1 - s^2 - t^2}, s, t)$$

$$\Phi_6(s,t) = (-\sqrt{1 - s^2 - t^2}, s, t)$$

Sphere as Union of 6 Coordinate Charts

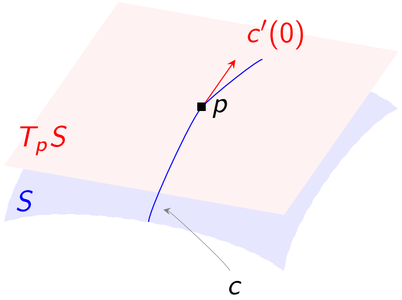


- Each $\Phi_k(D)$ is an open hemisphere
- Every point in S lies in one of the $\Phi_k(D)$, because
  - If $(x, y, z) \in S$, then at least one of the coordinates is nonzero
  - Say $x < 0$
  - Then $y^2 + z^2 = 1 - x^2 < 1$ and therefore $(y, z) \in D$ and

$$(x, y, z) = \Phi_6(y, z) \in \Phi_5(D)$$

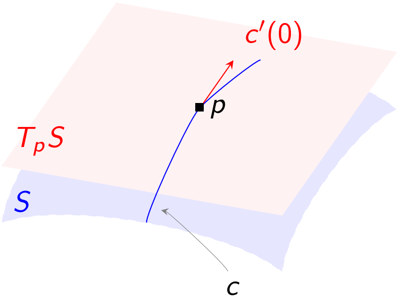
- In problem 4 of Homework 6, only two coordinate charts are used

Tangent Space at Point on Surface



- Let $S \subset \mathbb{A}^3$ be a C^1 surface and $p \in S$
- The tangent space at $p \in S$, denoted T_pS, is the set of all possible velocity vectors of C^1 curves in S that pass through p
- Since a curve in S is also a curve in $\mathbb{A}^3$, its velocity at p is in the tangent space $\mathbb{V}^3$
- Therefore, $T_pS \subset \mathbb{V}^3$

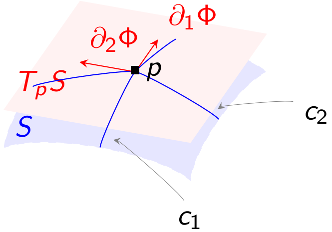
Tangent Space at a Point on a Surface



- Let $S \subset \mathbb{A}^3$ be a C^1 surface and $p \in S$
- A vector $v \in \mathbb{V}^3$ lies in the tangent space T_pS if there is a C^1 curve $c : (-\delta, \delta) \to S$ such that

$$c(0) = p \text{ and } c'(0) = v$$

Tangent Vectors with Respect to a Coordinate Chart



- Suppose $\Phi : D \to S$ is a coordinate chart with $\Phi(0,0) = p$
- The velocity vectors of the curves $c_1(t) = \Phi(t, 0)$ and $c_2(t) = \Phi(0, t)$ are the partial derivatives of Φ:

$$c_1'(0) = \partial_1\Phi(0,0) \text{ and } c_2'(0) = \partial_2\Phi(0,0),$$

- Given any vector $\langle v^1, v^2\rangle \in \mathbb{R}^2$, if

$$c(t) = \Phi(v^1 t, v^2 t),$$

then

$$c'(0) = \partial\Phi(0,0)(\langle v^1, v^2\rangle) = v^1\partial_1\Phi(0,0) + v^2\partial_2\Phi(0,0)$$

Tangent Space at point on a Surface is a 2D Vector Space

- The Jacobian at p is a rank 2 linear map

$$\begin{aligned} \partial\Phi(0,0) : \mathbb{R}^2 &\to T_pS \subset \mathbb{V}^3 \\ \langle v^1, v^2 \rangle &\mapsto v^1 \partial_1 \Phi(0,0) + v^2 \partial_2 \Phi(0,0) \end{aligned}$$

- In fact, it is a linear isomorphism, and T_pS is 2-dimensional
- Therefore, $\partial_1 \Phi$ and $\partial_2 \Phi$ are a basis of T_pS
- For convenience, we often write $\partial_1 = \partial_1 \Phi$ and $\partial_2 \Phi$

Tangent Space at a Point on Sphere

- Let $p_0 \in \mathbb{E}^3$, $R > 0$, and

$$S = \{p \in \mathbb{E}^3 \ : \ (p - p_0) \cdot (p - p_0) = R^2\}$$

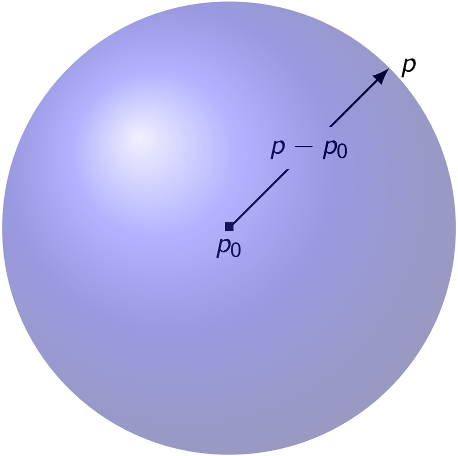
- Given $p \in S$, $v \in T_pS$, consider any C^1 curve $c : (-\delta, \delta) \to S$ such that $c(0) = p$ and $c'(0) = v$
- Since $(c(t) - p_0) \cdot (c(t) - p_0) = R^2$,

$$0 = \left.\frac{d}{dt}\right|_{t=0} (c(t) - p_0) \cdot (c(t) - p_0) = 2(c(0) - p_0) \cdot c'(t) = 2(p - p_0) \cdot v$$

- Therefore, any tangent vector $v \in T_pS$ is orthogonal to $p - p_0 \in \mathbb{V}^3$
- Since the space of all vectors in $\mathbb{V}^3$ that are orthogonal to a fixed nonzero vector $p - p_0$ is 2-dimensional, we conclude that

$$T_pS = \{v \in \mathbb{V}^3 \ : \ v \cdot (p - p_0) = 0\}$$

Tangent Space at a Point on a Sphere



Tangent Bundle

- Each $p \in S$ has its own tangent space T_pS
- Given two different points $p_1, p_2 \in S$, there is no direct relationship between their tangent spaces
- There is no natural way to move a vector $v \in T_{p_1}S$ to $T_{p_2}S$
  - Unless they are the same
- If $S \subset \mathbb{A}^3$ is an affine plane, then
  - $T_pS = T_qS$ for any $p, q \in S$
- The disjoint union of the tangent spaces of all points on a surface is called the tangent bundle,

$$T_*S = \coprod_{p \in S} T_pS$$