MATH-UA 377 Differential Geometry
Spherical curves
Spherical Crofton formula
Fenchel's Theorem
Fary-Milnor Theorem

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March 1, 2022

START RECORDING
LIVE TRANSCRIPTION

Space of Great Circles

- A great circle is the intersection of S with a plane through the origin
- Let $\mathcal{G}$ be the space of all great circles in S
- For each $p \in S$, there is a corresponding great circle

$$G_p = \{q \in S \ : \ (q - p_0) \cdot (p - p_0) = 0\}$$

- The map

$$\begin{aligned} g : S &\to \mathcal{G} \\ p &\mapsto G_p \end{aligned}$$

  is 2-to-1
- Observe that given $p \in S$,

$$p \in G_q \iff q \in G_p$$

- The area of a subset $\Omega \subset \mathcal{G}$ is equal to

$$\text{Area}_{\mathcal{G}}(\Omega) = \frac{1}{2}\text{Area}_S(g^{-1}(\Omega))$$

Spherical Crofton formula

- Let $c : [0, \ell] \to S$ be a unit speed curve
- For each $p \in S$, let

$$n_c(p) = \text{number of points in } G_p \cap c$$

- The integral

$$\int_S n_c(p)\, dA_S(p)$$

  is a measure of how many times great circles intersect the curve c
- One would expect longer curves to have more intersections
- The spherical Crofton formula confirms this:

$$\int_S n_c(p)\, dA_S(p) = 4L(c)$$

- This formula is useful for global theorems on closed space curves

Fenchel's Theorem (1929)

- If c is a closed curve in $\mathbb{E}^3$ with nonvanishing curvature κ

$$\int \kappa\sigma \, ds \geq 2\pi,$$

- Setup of proof
  - Use a unit speed parameterization $c : [0, \ell] \to \mathbb{E}^3$
  - Let (f_1, f_2, f_3) be the Frenet-Serret frame along c
  - Observe that $f_1 : [0, \ell] \to S$ is a curve on the unit sphere
  - By the first Frenet-Serret equation, the speed of the curve f_1 is

$$|\dot{f}_1| = |\kappa f_2| = \kappa$$

  - Therefore, the length of the spherical curve f_1 is

$$\int_{s=0}^{s=\ell} |\dot{f}_1| \, ds = \int_{s=0}^{s=\ell} \kappa(s) \, ds$$

  - On the other hand, by the spherical Crofton formula,

$$\int_{s=0}^{s=\ell} |\dot{f}| \, ds = \frac{1}{4} \int_{S^2} n_{f_1}(p) \, dA_S(p)$$

Proof of Fenchel's Theorem

- It suffices to prove that $n_{f_1}(p) \geq 2$ for all $p \in S$
- Suppose there is a great circle G_v such that $n_{f_1}(v) = 0$
- The curve f_1 must lie in one of the hemispheres with boundary G_v,

  $$H_v = \{u \in S \ : \ u \cdot v \geq 0\} \text{ or } H_{-v} = \{u \in S \ : \ u \cdot v \leq 0\}$$

  It follows that

  $$\begin{aligned} 0 &\leq \int_{s=0}^{s=\ell} v \cdot f_1 \, ds \\ &= \int_{s=0}^{s=\ell} v \cdot \dot{c} \, ds \\ &= \int_{s=0}^{s=\ell} \frac{d}{ds}(v \cdot (c(s) - c(0))) \, ds \\ &= v \cdot (c(\ell) - c(0)) = 0 \end{aligned}$$

- It follows that $v \cdot f_1(s) = 0$ for all $s \in [0, \ell]$, wihch implies the curve is planar
- The curve f_1 must therefore cross any great cricle

Fary-Milnor Theorem

Theorem

(Fary, 1949 and Milnor, 1950) If an embedded closed curve c is knotted, then

$$\int_c \kappa \, ds \geq 4\pi$$

Spherical Curve

- Given $p_0 \in \mathbb{E}^3$, let S be the unit sphere centered at p_0

$$S = \{p \in \mathbb{E}^3 \ : \ |p - p_0| = 1\}$$

- Consider a unit speed curve $c : [0, \ell] \to S$
- Given an orientation on $\mathbb{E}^3$, there is a unique positively oriented frame (e_1, e_2, e_3) on the curve such that

$$\begin{aligned} e_1 &= \dot{c} \\ e_3 &= p - p_0 \end{aligned}$$

- e_3 is normal to S, and e_1, e_2 are tangent to S
- From HW4, problem 4.3, we know there is a function $\kappa_g : [0, \ell] \to \mathbb{R}$ such that

$$\frac{d}{dt}\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} 0 & -\kappa_g & 1 \\ \kappa_g & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

- κ_g is called the geodesic curvature

Proof of Spherical Crofton Formula

- Consider the map

$$F : [0, \ell] \times [0, 2\pi] \to S$$
$$(t, \theta) \mapsto p_0 + e_1(t) \cos\theta + e_2(t) \sin\theta \in G_{c(t)}$$

- For a fixed t, the map $\theta \mapsto F(t, \theta)$ is a parameterization of the great circle $G_{c(t)}$
- On the other hand, $c(t) \in G_q \iff q \in G_{c(t)}$
- Therefore, $\theta \mapsto F(t, \theta)$ is a parameterization of all great circles that contain $c(t)$
- It follows that for each $p \in S$,

$$n_c(p) = \frac{1}{2} \text{ number of points in } F^{-1}(p)$$

- If we view F as a multi-valued parameterization of S, then

$$\int_{[0,\ell]\times[0,2\pi]} F^* \, dA_S = 2 \int_{\mathcal{G}} n_c(p) \, dA_{\mathcal{G}} = \int_S n_c(p) \, dA_S,$$

where

$$F^* dA_S = |F_t \times F_\theta| \, dt \, d\theta$$

Partial derivatives of F

- Recall that

$$\begin{aligned} \dot{e}_1 &= e_2\kappa_g - e_3 \\ \dot{e}_2 &= -e_1\kappa_g \end{aligned}$$

- Since $F(t,\theta) = p_0 + e_1(t)\cos\theta + e_2(t)\sin\theta$,

$$\begin{aligned} F_t &= (e_2\kappa_g - e_3)\cos\theta + -e_1\kappa_t\sin\theta \\ &= \kappa_g(-e_1\sin\theta + e_2\cos\theta) - e_3\cos\theta \\ F_\theta &= -e_1\sin\theta + e_2\cos\theta \\ F_t \times F_\theta &= \cos\theta(e_3 \times (e_1\sin\theta - e_2\cos\theta)) \\ &= \cos\theta(e_2\sin\theta + e_1\cos\theta) \\ |F_t \times F_\theta| &= |\cos\theta| \end{aligned}$$

Spherical Crofton Formula

- Therefore,

$$\begin{aligned}\int_S n_c(p)\, dA_p &= \int_{[0,\ell]\times[0,2\pi]} F^* dA \\ &= \int_{t=0}^{t=\ell} \int_{\theta=0}^{\theta=2\pi} |\cos\theta|\, d\theta\, dt \\ &= 4\ell \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos\theta\, d\theta \\ &= 4\ell\end{aligned}$$

- The spherical Crofton formula states that the length of a curve c in S is equal to

$$\ell = \frac{1}{4}\int_S n_c(p)\, dA_S(p)$$