MATH-UA 377 Differential Geometry
Open set in $\mathbb{R}^2$
C^1 Map and its Partial Derivatives
Parameterized Surface in $\mathbb{V}^3$

Deane Yang

Courant Institute of Mathematical Sciences
New York University

March 1, 2022

START RECORDING
LIVE TRANSCRIPTION

Topology of $\mathbb{R}^2$

- We will denote a point in $\mathbb{R}^2$ sometimes by (x^1, x^2), sometimes (x, y), sometimes (s, t), sometimes (u, v), and sometimes something completely different
- An open ball or disk in $\mathbb{R}^2$ with center $(x_0^1, x_0^2) \in \mathbb{R}^2$ and radius $r > 0$ is the set

$$B((x_0^1, x_0^2), r) = \{(x^1, x^2) \in \mathbb{R}^2 \ : \ (x^1 - x_0^1)^2 + (x^2 - x_0^2)^2 < r^2\}$$

- A set $O^2 \subset \mathbb{R}^2$ is open if for each point $(x_0^1, x_0^2) \in O^2$, there is a ball $B((x_0^1, x_0^2), r) \subset O^2$
  - $r > 0$ might have to be very small

C^1 Map and Its Partial Derivatives

- A map $\Phi : O^2 \to \mathbb{A}^3$ is C^1 if for each $B((x_0^1, x_0^2), r) \subset O^2$, the maps

$$
\begin{aligned}
c_1 : (-r, r) &\to \mathbb{A} \\
t &\mapsto \Phi(x_0^1 + t, x_0^2) \\
c_2 : (-r, r) &\to \mathbb{A} \\
t &\mapsto \Phi(x_0^1, x_0^2 + t)
\end{aligned}
$$

  are C^1
- The partial derivatives of the map Φ at a point $(x_0, y_0) \in O^2$ are defined to be

$$
\begin{aligned}
\partial_1 \Phi(x_0^1, x_0^2) &= \left.\frac{d}{dt}\right|_{t=0} \Phi(x_0^1 + t, x_0^2) \in \mathbb{V}^3 \\
\partial_2 \Phi(x_0^1, x_0^2) &= \left.\frac{d}{dt}\right|_{t=0} \Phi(x_0^1, x_0^2 + t) \in \mathbb{V}^3
\end{aligned}
$$

- Each partial derivative is a map $\partial_k \Phi : O^2 \to \mathbb{V}^3$

Jacobian of a map $\Phi : O^2 \to \mathbb{A}^3$

- The Jacobian of a C^1 map $\Phi : O^2 \to \mathbb{A}^3$ is defined to be the matrix of partial derivatives

$$\partial\Phi = \begin{bmatrix} \partial_1\Phi & \partial_2\Phi \end{bmatrix}$$

- For each $(x_0^1, x_0^2) \in O^2$, the Jacobian defines a linear map

$$\begin{aligned} \partial\Phi(x_0^1, x_0^2) : \widehat{R}^2 &\to \mathbb{V}^3 \\ v = \langle v^1, v^2 \rangle &\mapsto \partial\Phi(x_0^1, x_0^2)v \\ &= \begin{bmatrix} \partial_1\Phi & \partial_2\Phi \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\ &= v^1\partial_1\Phi(x_0^1, x_0^2) + v^2\partial_2\Phi(x_0^1, x_0^2) \end{aligned}$$

Jacobian of a map $\Phi : O^2 \to \mathbb{R}^3$

- A map $\Phi : O^2 \to \mathbb{R}^3$ can be written as

$$\Phi(x^1, x^2) = (\Phi^1(x^1, x^2), \Phi^2(x^1, x^2), \Phi^3(x^1, x^2)),$$

  where each Φ^k is a scalar function on O^2
- The Jacobian of Φ can be written as

$$\partial\Phi = \begin{bmatrix} \partial_1\Phi & \partial_2\Phi \end{bmatrix} = \begin{bmatrix} \partial_1\Phi^1 & \partial_2\Phi^1 \\ \partial_1\Phi^2 & \partial_2\Phi^2 \\ \partial_1\Phi^3 & \partial_2\Phi^3 \end{bmatrix}$$

- For each $(x_0^1, x_0^2) \in O^2$, the Jacobian defines a linear map

$$\begin{aligned} \partial\Phi(x_0^1, x_0^2) : \widehat{R}^2 &\to \mathbb{V}^3 \\ v = \langle v^1, v^2 \rangle &\mapsto \partial\Phi(x_0^1, x_0^2)v \\ &= \begin{bmatrix} \partial_1\Phi^1 & \partial_2\Phi^1 \\ \partial_1\Phi^2 & \partial_2\Phi^2 \\ \partial_1\Phi^3 & \partial_2\Phi^3 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \\ &= v^1\partial_1\Phi(x_0^1, x_0^2) + v^2\partial_2\Phi(x_0^1, x_0^2) \end{aligned}$$

Nondegenerate Map

- A C^1 map $\Phi : O \to \mathbb{A}^3$ is *nondegenerate* if for each $(x^1, x^2) \in O$, its Jacobian, which is a linear map
$$\partial\Phi(x^1, x^2) : \widehat{R}^2 \to \mathbb{V}^3,$$
has maximal rank (equal to 2)
- Equivalently, Φ is nondegenerate if for each $(x^1, x^2) \in O$, the vectors
$$\partial_1\Phi(x^1, x^2), \partial_2\Phi(x^1, x^2) \in \mathbb{V}^3$$
are linearly independent
- Equivalently, Φ is nondegenerate if for each $(x^1, x^2) \in O$, the image of the linear map
$$\partial\Phi(x^1, x^2) : \widehat{R}^2 \to \mathbb{V}^3,$$
is a 2-dimensional subspace of $\mathbb{V}^3$

Parameterized Surface in $\mathbb{A}^3$

- Recall that a parameterized curve is a C^1 map $c : I \to \mathbb{A}^3$ that has nonzero speed $\dot{c}(t)$ for every $t \in I$
- The 2-dimensional analogue of nonzero speed is nondengeracy
- The 2-dimensional analogue of an interval in $\mathbb{R}$ is an open set in $\mathbb{R}^2$
- A parameterized surface is a nondegenerate injective map C^1 map $\Phi : O \to \mathbb{A}^3$
- Example: Paraboloid

$$\Phi(x, y) = (x, y, x^2 + y^2)$$

- Bad parameterization of paraboloid

$$\Phi(s, t) = (s^3, t^3, s^6 + t^6)$$

- Bad surface: Cone

$$\Phi(u, v) = (u^3, v^3, (u^6 + v^6)^{1/2})$$