MATH-UA 377 Differential Geometry
Frenet-Serret frame and equations in $\mathbb{E}^3$
Spherical curves
Space of great circles

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START RECORDING
LIVE TRANSCRIPTION

Key Trick

- Suppose $v, w : I \to \mathbb{V}$
- If $|v|$ is constant, then so is $|v|^2 = v \cdot v$
- Differentiating,

$$0 = \frac{d}{dt}(v \cdot v) = 2v \cdot \dot{v} \implies v \cdot \dot{v} = 0$$

- If $v \cdot w$ is constant, then

$$0 = \frac{d}{dt}(v \cdot w) = \dot{v} \cdot w + v \cdot \dot{w}$$

- Therefore, if $v \cdot w$ is constant, then

$$\dot{v} \cdot w = -v \cdot \dot{w}$$

Frenet-Serret Frame for Parameterized Curve in $\mathbb{E}^3$

- Fix an orientation on $\mathbb{E}^3$
- Let $c : I \to \mathbb{E}^3$ be a C^2 parameterized curve
- Assume, for any $t \in I$, $c'(t) \neq 0$ and set

$$\sigma = |c'| \text{ and } f_1 = \frac{c'}{|c'|}$$

- Assume $f_1{}'(t) \neq 0$ and recall that since $f_1 \cdot f_1 = 1$,

$$f_1 \cdot f_1{}' = 0$$

- Let $f_2(t)$ be the unit vector in same direction as $f_1{}'$
- Let $f_3(t)$ be the unique vector such that $(f_1(t), f_2(t), f_3(t))$ is a positively oriented orthonormal frame
- This is called the Frenet-Serret frame of the curve c
- It requires that $f_1{}' \neq 0$, which implies the curve is always changing direction

Curvature and Torsion

- Since $f_2(t)$ points in the same direction as ${f_1}'(t)$, there is a positive scalar function $\kappa : I \to (0, \infty)$ such that

$$ {f_1}'(t) = \sigma\kappa(t) f_2(t) $$

- κ is called curvature
  - Measures rate of change of direction of curve
- Let

$$ \tau = \frac{f_3 \cdot {f_2}'}{\sigma} $$

- τ is called torsion
  - Measures how fast the curve is twisting out of the plane spanned by f_1 and f_2

Deriving the Frenet-Serret equations

- ${f_1}' = \sigma\kappa f_2$
- Since $f_1 \cdot f_2 = 0$, $f_2 \cdot f_2 = 1$, and $f_3 \cdot f_2 = 0$,

$$
\begin{aligned}
f_1 \cdot {f_2}' &= -{f_1}' \cdot f_2 = -\sigma\kappa \\
f_2 \cdot {f_2}' &= 0 \\
f_3 \cdot {f_2}' &= \sigma\tau,
\end{aligned}
$$

- It follows that

$$
{f_2}' = -\sigma\kappa f_1 + \sigma\tau f_3
$$

- Since $f_1 \cdot f_3 = 0$, $f_2 \cdot f_3 = 0$, and $f_3 \cdot f_3 = 1$,

$$
\begin{aligned}
f_1 \cdot {f_3}' &= -{f_1}' \cdot f_3 = 0 \\
f_2 \cdot {f_3}' &= -{f_2}' \cdot f_3 = -\sigma\tau \\
f_3 \cdot {f_3}' &= 0
\end{aligned}
$$

- It follows that

$$
{f_3}' = -\sigma\tau f_2
$$

Frenet-Serret Equations for Curve in $\mathbb{E}^3$

- As a system of equations,

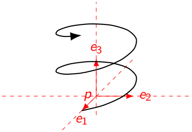
$$\begin{aligned} c' &= \sigma f_1 \\ {f_1}' &= \sigma(\kappa f_2) \\ {f_2}' &= \sigma(-\kappa f_1 + \tau f_3) \\ {f_3}' &= \sigma(-\tau f_2) \end{aligned}$$

- Using matrices,

$$\frac{d}{dt}\begin{bmatrix} c & f_1 & f_2 & f_3 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \sigma \begin{bmatrix} 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & -\tau \\ 0 & 0 & \tau & 0 \end{bmatrix}$$

- A curve is planar if and only if its torsion is always zero

Example: Helix in $\mathbb{R}^3$



$$c(t) = p + e_1 \cos t + e_2 \sin t + e_3 t$$

$$\dot{c}(t) = -e_1 \sin t + e_2 \cos t + e_3 \implies \sigma = |\dot{c}| = \sqrt{2}$$

$$f_1 = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1\rangle \implies \dot{f}_1 = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0\rangle$$

$$f_2 = \frac{\dot{f}_1}{|\dot{f}_1|} = \langle -\cos t, -\sin t, 0\rangle \implies \dot{f}_2 = \langle \sin t, -\cos t, 0\rangle$$

$$f_3 = f_1 \times f_2 = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle$$

$$\kappa = \frac{|\dot{f}_1|}{\sigma} = \frac{1}{2} \text{ and } \tau = \frac{f_3 \cdot \dot{f}_2}{\sigma} = \frac{1}{2}$$

Existence and Uniqueness Theorem

Given

- A C^1 function $\sigma : [0, T] \to (0, \infty)$
- Continuous functions

$$\kappa : [0, T] \to (0, \infty)$$
$$\tau : [0, T] \to \mathbb{R}$$

- A point $p \in \mathbb{E}^3$ and a positively oriented orthonormal basis (u_1, u_2, u_3) of $\mathbb{V}^3$,

there is a unique C^3 curve $c : [0, T] \to \mathbb{E}^3$ with speed σ, curvature κ, torsion τ, and Frenet-Serret frame $f_1, f_2, f_3)$ such that

$$c(0) = p$$
$$f_1(0) = u_1$$
$$f_2(0) = u_2$$
$$f_3(0) = u_3$$

Curvature and Torsion Uniquely Determine Shape of Curve

- Suppose c_1 and c_2 are parameterized curves with the same speed, curvature, torsion functions
- If F_1 is the Frenet-Serret frame of c_1 and F_2 is the Frenet-Serret frame of c_2, there is a unique rotation map $R : \mathbb{V}^3 \to \mathbb{V}^3$ such that $F_2(0) = R(F_1(0))$
- If $R : \mathbb{E}^3 \to \mathbb{E}^3$ is the rigid motion given by

  $$R(p) = c_2(0) + L(p - c_1(0)),$$

  then
  - $(R \circ c_1)(0) = c_2(0)$
  - The curve $R \circ c_1$ has the same speed, curvature, and torsion functions as c_1 and c_2
  - The Frenet-Serret frames of $R \circ c_1$ and c_2 are equal at $t = 0$
  - By the uniqueness theorem, $c_2 = R \circ c_1$

Spherical Curve

- Given $p_0 \in \mathbb{E}^3$, let S be the unit sphere centered at p_0

$$S = \{p \in \mathbb{E}^3 \ : \ |p - p_0| = 1\}$$

- Consider a unit speed curve $c : [0, \ell] \to S$
- Given an orientation on $\mathbb{E}^3$, there is a unique positively oriented frame (e_1, e_2, e_3) on the curve such that

$$\begin{aligned} e_1 &= \cdot c \\ e_3 &= p - p_0 \end{aligned}$$

- e_3 is normal to S, and e_1, e_2 are tangent to S
- From HW4, problem 4.3, we know there is a function $\kappa_g : [0, \ell] \to \mathbb{R}$ such that

$$\frac{d}{dt}\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} 0 & -\kappa_g & 1 \\ \kappa_g & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

- κ_g is called the geodesic curvature

Space of Great Circles

- A great circle is the intersection of S with a plane through the origin
- Let $\mathcal{G}$ be the space of all great circles in S
- For each $p \in S$, there is a corresponding great circle

$$G_p = \{q \in S \ : \ (q - p_0) \cdot (p - p_0) = 0\}$$

- The map

$$\begin{aligned} g : S &\to \mathcal{G} \\ p &\mapsto G_p \end{aligned}$$

  is 2-to-1
- Observe that given $p \in S$,

$$p \in G_q \iff q \in G_p$$

- The area of a subset $\Omega \subset \mathcal{G}$ is equal to

$$\text{Area}_{\mathcal{G}}(\Omega) = \frac{1}{2}\text{Area}_S(g^{-1}(\Omega))$$