MATH-UA 377 Differential Geometry
Curves in Euclidean 3-Space

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START RECORDING
LIVE TRANSCRIPTION

Frenet-Serret Frame for Parameterized Curve in $\mathbb{E}^3$

- Fix an orientation on $\mathbb{E}^3$
- Let $c : I \to \mathbb{E}^3$ be a C^2 parameterized curve
- Assume, for any $t \in I$, $c'(t) \neq 0$ and set

$$\sigma = |c'|$$
$$f_1 = \frac{c'}{|c'|}$$

- Assume $f_1{}'(t) \neq 0$ and let $f_2(t)$ be the unit vector in same direction
- Since $f_1{}' \cdot f_1 = 0$, $f_2 \cdot f_1 = 0$
- For each $t \in I$, there is a unique vector $f_3(t)$ such that $F(t) = (f_1(t), f_2(t), f_3(t))$ is an oriented orthonormal frame
- This is called the Frenet-Serret frame of the curve c
- It requires that $f_1{}' \neq 0$, which implies the curve is always changing direction

First Two Frenet-Serret Equations

- Since $f_2(t)$ points in the same direction as $f_1{}'(t)$, there is a positive scalar function $\kappa : I \to (0, \infty)$ such that

$$f_1{}'(t) = \sigma\kappa(t) f_2(t)$$

- Since $f_2{}' \cdot f_2 = 0$, there are scalar functions α and τ such that

$$f_2{}' = \sigma(\alpha f_1 + \tau f_3)$$

- Since

$$0 = (f_1 \cdot f_2)' = f_1 \cdot f_2{}' + f_1{}' \cdot f_2 = \sigma(\alpha + \kappa),$$

  it follows that $\alpha = -\kappa$
- κ is the rate of change of the direction of the curve and is called the **curvature**
- τ is the rate of change of f_2 in the direction of f_3 and is called the **torsion**
  - Torsion measures how fast the curve is twisting out of the plane spanned by f_1 and f_2