MATH-UA 377 Differential Geometry
Winding and Rotation Numbers of a Closed Curve

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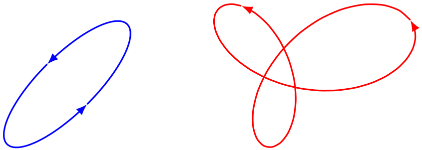
LIVE TRANSCRIPTION

Smooth Immersed Curves



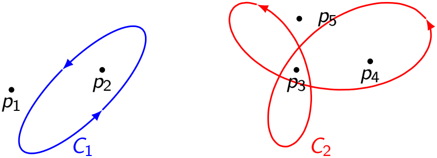
- Recall that we have defined a smooth parameterized curve in an affine space $\mathbb{A}$ to be a smooth map

$$c : I \to \mathbb{A}$$

where $\dot{c}(t) \neq 0$ for all $t \in I$

- Note that we allow the curve to intersect itself
- Such curves are also called immersed curves
- If a curve does not intersect itself, it is called an embedded curve

Winding versus Rotation Number of a Closed Planar Curve



- Winding number $W(p, C)$ of a closed planar curve $C \subset \mathbb{A}^2$ around a point $p \notin C$ is the number of times the curve goes counterclockwise around p

$$W(p_1, C_1) = W(p_5, C_2) = 0$$
$$W(p_2, C_1) = W(p_4, C_2) = 1$$
$$W(p_3, C_2) = 2$$

- Rotation number $R(C)$ is the number of times the unit tangent vector rotates counterclockwise around the circle

$$R(C_1) = 1$$
$$R(C_2) = 2$$

"Obvious" Facts about the Winding and Rotation Numbers

- The winding number $W(C, p)$
  - Depends on where p lies relative to the curve
  - Equals zero if p lies outside the curve completely
  - If p_1 and p_2 are points that can be connected by a curve that does not cross C, then

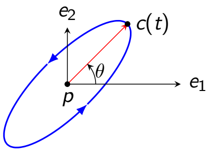
$$W(C, p_1) = W(C, p_2)$$

  - If a curve C_1 can be continuously deformed through a family of closed curves into another curve C_2 without any of the curves crossing p, then

$$W(C_1, p) = W(C_2, p)$$

- The rotation number
  - Remains unchanged under any smooth deformation of the curve

Derivative of Polar Angle



- A smooth curve $c : [0, T] \to \mathbb{E}^2$ can be written using polar coordinates relative to a point p not on the curve as

$$c(t) = p + e_1 x(t) + e_2 y(2) = p + r(t)(e_1 \cos(\theta(t)) + e_2 \sin(\theta(t))),$$

  where $r(t)$ is always nonzero
- Differentiating this, we get

$$\begin{aligned} e_1 \dot{x} + e_2 \dot{y} &= \dot{r}(e_1 \cos\theta + e_2 \sin\theta) + \dot{\theta}(-e_1 r \sin\theta + e_2 r \cos\theta) \\ &= \frac{\dot{r}}{r}(e_1 x + e_2 y) + \dot{\theta}(-e_1 y + e_2 x) \end{aligned}$$

- Therefore,

$$\dot{\theta} = \frac{-y\dot{x} + x\dot{y}}{x^2 + y^2} \text{ and } \theta(T) - \theta(0) = \int_{t=0}^{t=T} \frac{-y\dot{x} + x\dot{y}}{x^2 + y^2}\, dt$$

Winding Number of a Closed Curve

- If $c : [0, T] \to \mathbb{E}^2$ is a closed curve and p does not lie on the curve, then

$$
\begin{aligned}
c(0) = c(T) &\implies x(0) = x(T) \text{ and } y(0) = y(T) \\
&\implies r(0) = r(T) \text{ and } \\
&\quad\ \theta(T) - \theta(0) = 2\pi k, \text{ for some integer } k
\end{aligned}
$$

- Therefore,

$$
\frac{1}{2\pi}\int_{t=0}^{t=T} \frac{-y\dot{x} + x\dot{y}}{x^2 + y^2}\, dt = \frac{1}{2\pi}\int_{t=0}^{t=T} \dot{\theta}(t)\, dt = \theta(T) - \theta(0) = k
$$

- Equivalently, the line integral

$$
\frac{1}{2\pi}\int_C \frac{-y\, dx + x\, dy}{x^2 + y^2}
$$

is always an integer and equal to the winding number

Winding Number is a Topological Invariant

- Suppose $c_\delta : [0,1] \to \mathbb{E}^2$ is a continuous family of closed curves, parameterized by $0 \leq \delta \leq 1$
- In other words, for each $0 \leq \delta \leq 1$, the curve c_δ satisfies

$$c_\delta(0) = c_\delta(1)$$

- If we define the polar angle θ such that for each $0 \leq \delta \leq 1$,

$$\theta_\delta(0) = 0$$

  then

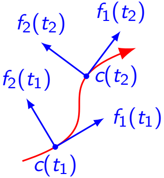
$$\theta_\delta(1) = 2\pi k_\delta$$

- On the other hand,

$$\theta_\delta(1) - \theta_\delta(0) = \int_{t=0}^{t=1} \frac{-y_\delta \dot{x}_\delta + x_\delta \dot{y}_\delta}{x_\delta^2 + y_\delta^2}\, dt$$

  is a continuous function of δ
- Therefore, the winding number $W(C_\delta, p) = k_\delta$ is a constant independent of δ

Frenet-Serret Frame and Equations for Parameterized Curve in $\mathbb{E}^2$



- The Frenet-Serret frame for a parameterized curve $c : I \to \mathbb{E}^2$ is an oriented orthonormal frame $F = (f_1, f_2)$ along c such that

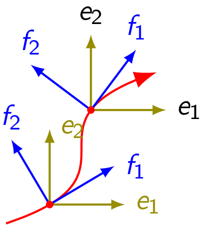
$$c' = \sigma f_1$$

- The Frenet-Serret equations are

$$\frac{1}{\sigma}\frac{d}{dt}\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix}\begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix},$$

  where κ is the curvature function

Rotation Angle of a Parameterized Curve



- Fix an orthonormal basis (e_1, e_2) of $\mathbb{V}^2$
- Consider a curve $c : I \to \mathbb{E}^2$ with Frenet-Serret frame (f_1, f_2)
- The counterclockwise angle ϕ from e_1 to f_1 satisfies

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Curvature is Normalized Rate of Change of Angle

- On one hand, the Frenet-Serret equations say

$$\frac{d}{dt}\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix}\begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}\sigma$$

- On the other hand,

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} f_1 & f_2 \end{bmatrix} &= \begin{bmatrix} e_1 & e_2 \end{bmatrix}\begin{bmatrix} -\sin\phi & -\cos\phi \\ \cos\phi & -\sin\phi \end{bmatrix}\dot{\phi} \\ &= \begin{bmatrix} -e_1\sin\phi + e_2\cos\phi & -e_1\cos\phi - e_2\sin\phi \end{bmatrix}\dot{\phi} \\ &= \begin{bmatrix} f_2 & -f_1 \end{bmatrix}\dot{\phi} \\ &= \begin{bmatrix} f_1 & f_2 \end{bmatrix}\begin{bmatrix} 0 & -\dot{\phi} \\ \dot{\phi} & 0 \end{bmatrix}\end{aligned}$$

- Therefore,

$$\kappa = \frac{\dot{\phi}}{\sigma} \text{ or } \dot{\phi} = \sigma\kappa$$

Rotation Number of a Smooth Closed Curve

- If a curve $c : [0, T] \to \mathbb{A}^2$ is closed, then

$$c(0) = c(T)$$

- If a closed curve is smooth and oriented in the direction $\dot{c}$, then since $\dot{c}(0)$ and $\dot{c}(T)$ have the same orientation, they have to point in the same direction
- Therefore,

$$\phi(T) - \phi(0) = 2\pi k,$$

  where k is the rotation number of C
- Since

$$\dot{\phi} = \kappa\sigma,$$

  the rotation number of C is equal to

$$R(C) = \frac{1}{2\pi} \int_{t=0}^{t=T} \kappa(t)\sigma(t)\, dt$$

Rotation Number is a Topological Invariant

- If c_δ is a continuous family of curves parameterized by $\delta \in [0,1]$ such that the curvature function κ_δ and speed function σ_δ are continuous functions of δ, then

$$R(C_\delta) = \frac{1}{2\pi} \int_{t=0}^{t=T} \kappa_\delta(t)\sigma_\delta(t)\, dt$$

  is a continuous function of δ
- Since $R(C_\delta)$ is an integer, it must therefore be constant