MATH-UA 377 Differential Geometry
Orientation of Affine Space
Orientation and Curvature of a Curve in $\mathbb{E}^2$
Frenet-Serret Frame and Equations

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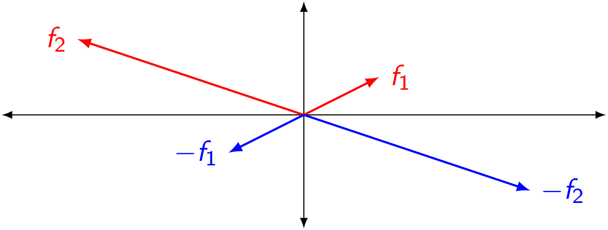
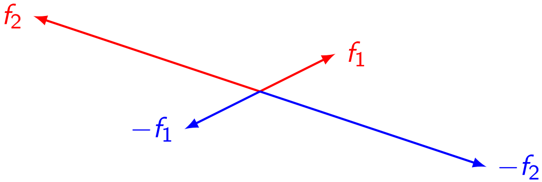
LIVE TRANSCRIPTION

Oriented Basis of $\mathbb{R}^2$

- A basis (f_1, f_2) of $\mathbb{R}^2$ is called *positively oriented*, if the vector f_2 is f_1 rotated counterclockwise by an angle less than π
- Otherwise, the basis has negative orientation
- If (f_1, f_2) has positive orientation, then
  - (f_2, f_1), $(-f_1, f_2)$, $(f_1, -f_2)$ have negative orientation
  - $(-f_1, -f_2)$, $(f_2, -f_1)$, $(-f_2, f_1)$ have positive orientation



Facts About Oriented Bases of $\mathbb{R}^2$



- The standard basis $(\langle 1,0\rangle, \langle 0,1\rangle)$ has positive orientation
- If E and F are bases, then there is an invertible matrix M such that $F = EM$.
    - If $\det M > 0$, then E and F have the same orientation
    - If $\det M < 0$, then they have opposite orientations

Oriented Bases of an Abstract Vector Space $\mathbb{V}^m$

- There is no standard basis of $\mathbb{V}^2$
- If E and $F = EM$ are bases, then we say they have the same orientation if $\det M > 0$.
- The set $\mathcal{B}$ of all bases of $\mathbb{V}^2$ is the union of two disjoint subsets,

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2,$$

  where E and $F = EM$ are in the same subset if and only if $\det M > 0$ and in different subsets if and only if $\det M < 0$
- An orientation on $\mathbb{V}^m$ is choosing one of the two subsets and saying that the bases in that subset have positive orientation
- An orientation of an affine space is defined to be an orientation of its tangent space

Oriented Bases of $\mathbb{V}^2$

- Suppose the basis

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix}$$

  has positive orientation

- The following bases also have positive orientation:

$$\begin{bmatrix} -f_1 & -f_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -f_2 & f_1 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- The following bases have negative orientation:

$$\begin{bmatrix} f_1 & -f_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} f_2 & f_1 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

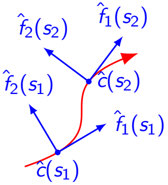
Orientation of a Curve in $\mathbb{A}^m$

- Suppose $c : [a, b] \to \mathbb{A}^m$ is a parameterized curve
- An orientation of c is a choice of direction along the curve, either $\dot{c}$ or $-\dot{c}$
- This is equivalent to choosing which endpoint, $c(a)$ or $c(b)$, is the start of the curve and which is the end
- Standard orientation:
  - The direction $\dot{c}$
  - If $a < b$, then $c(a)$ is the start and $c(b)$ is the end

Frenet-Serret-Frame along Unit Speed Curve in $\mathbb{E}^2$

- Choose an orientation on $\mathbb{E}^2$
- Suppose $\hat{c} : I \to \mathbb{E}^2$ is a unit speed curve with the standard orientation
- For each $s \in I$,
  - Let $\hat{f}_1(s) = \hat{c}'(s)$
  - There is a unique vector $\hat{f}_2(s)$ such that $(\hat{f}_1(s), \hat{f}_2(s))$ is a positively oriented orthonormal basis of $\mathbb{V}^2$
  - $(\hat{f}_1, \hat{f}_2)$ is called an *adapted positively oriented orthonormal frame along the curve* $\hat{c}$
  - $(\hat{f}_1, \hat{f}_2)$ is also called a Frenet-Serret frame

Oriented Curvature of an Oriented Unit Speed Curve in $\mathbb{E}^2$



- Let $(\hat{f}_1, \hat{f}_2)$ be a Frenet-Serret frame along an oriented unit speed curve $c : I \to \mathbb{E}^2$
- Recall that since $\hat{f}_1 \cdot \hat{f}_1 = 1$, $0 = \frac{d}{ds}(\hat{f}_1 \cdot \hat{f}_1) = 2\hat{f}_1{}' \cdot \hat{f}_1$
- $\hat{f}_1{}'(s)$ must point in the same or opposite direction to $\hat{f}_2(s)$
- Therefore, there is a scalar function $\hat{\kappa}$ such that

$$\hat{f}_1{}'(s) = \hat{\kappa}(s)\hat{f}_2(s)$$

- $\hat{\kappa}$ is called the oriented curvature function
- In picture above, $\hat{\kappa}(s_1) > 0$ and $\hat{\kappa}(s_2) < 0$

Frenet-Serret Equations of Unit Speed Curve in $\mathbb{E}^2$

- Let $(\hat{f}_1, \hat{f}_2)$ be a Frenet-Serret frame along an oriented unit speed curve $c : I \to \mathbb{E}^2$
- If we differentiate the equations

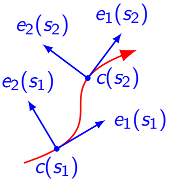
$$\hat{f}_1 \cdot \hat{f}_1 = \hat{f}_2 \cdot \hat{f}_2 = 1 \text{ and } \hat{f}_1 \cdot \hat{f}_2,$$

  we get

$$\begin{aligned} \hat{f}_1{}' \cdot \hat{f}_1 &= 0 \\ \hat{f}_2{}' \cdot \hat{f}_2 &= 0 \\ \hat{f}_1{}' \cdot \hat{f}_2 + \hat{f}_1 \cdot \hat{f}_2{}' &= 0 \end{aligned}$$

- By the definition of curvature and equations above,
  - $\hat{f}_1{}' = \hat{\kappa}\hat{f}_2$
  - $\hat{f}_2{}' = a\hat{f}_1$ for some scalar function a
  - $(\hat{\kappa}\hat{f}_2) \cdot \hat{f}_2 + \hat{f}_1 \cdot (a\hat{f}_1) = 0$, which implies $a = -\hat{\kappa}$

Frenet-Serret Equations of Unit Speed Curve in $\mathbb{E}^2$



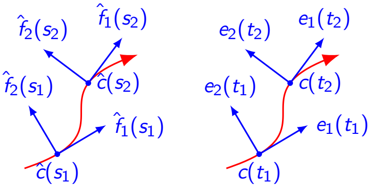
- Let $(\hat{f}_1, \hat{f}_2)$ be a Frenet-Serret frame along an oriented unit speed curve $c : I \to \mathbb{E}^2$
- The Frenet-Serret equations for a unit speed curve in $\mathbb{E}^2$ are

$$\hat{f}_1{}' = \hat{\kappa}\hat{f}_2$$
$$\hat{f}_2{}' = -\hat{\kappa}\hat{f}_1$$

- Equivalently

$$\frac{d}{ds}\begin{bmatrix}\hat{f}_1 & \hat{f}_2\end{bmatrix} = \begin{bmatrix}\hat{f}_1 & \hat{f}_2\end{bmatrix}\begin{bmatrix}0 & -\hat{\kappa}\\ \hat{\kappa} & 0\end{bmatrix}$$

Frenet-Serret Frame for Parameterized Curve



- Let $\mathbb{E}^2$ be oriented Euclidean space
- The Frenet-Serret frame of a C^2 parameterized curve $c : I \to \mathbb{E}^2$ is the unique positively oriented orthonormal frame (f_1, f_2) such that

$$f_1 = \frac{c'}{|c'|}$$

- If $\hat{c}$ is the unit speed parameterization of the same curve and $\widehat{E} = (\hat{f}_1, \hat{f}_2)$ is the Frenet-Serret frame for $\hat{c}$, then

$$c(t) = \hat{c}(s(t)) \text{ and } F(t) = \widehat{F}(s(t)),$$

where $s(t)$ is the arclength function

Frenet-Serret Equations for Parameterized Curve

- Since $s'(t) = \sigma(t)$ and using the chain rule,

$$\begin{aligned}
c'(t) &= \hat{c}'(s(t))s'(t) \\
&= \sigma(t)\hat{f}_1(s(t)) \\
&= \sigma(t)f_1(t) \\
{f_1}'(t) &= \hat{f_1}'(s(t))s'(t) \\
&= \sigma(t)\hat{\kappa}(s(t))\hat{f}_2(s(t)) \\
&= \sigma(t)\kappa(t)f_2(t) \\
{f_2}'(t) &= \hat{f_2}'(s(t))s'(t) \\
&= -\sigma\hat{\kappa}(s(t))\hat{f}_1(s(t)) \\
&= -\sigma\kappa(t)f_1(t),
\end{aligned}$$

where $\kappa(t) = \hat{\kappa}(s(t))$ is the curvature at $c(t) = \hat{c}(s(t))$

Frenet-Serret Frame and Equations for Parameterized Curve in $\mathbb{E}^2$

- The Frenet-Serret frame for a parameterized curve $c : I \to \mathbb{E}^2$ is an adapted oriented orthonormal frame $E = (f_1, f_2)$ along c such that
$$c' = \sigma f_1$$
- The Frenet-Serret equations are
$$\frac{1}{\sigma}\frac{d}{dt}\begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 \end{bmatrix}\begin{bmatrix} 0 & -\kappa \\ \kappa & 0 \end{bmatrix}$$

Example: Frenet-Serret Frame of a Circle

- Given a point $p \in \mathbb{E}^2$ and an orthonormal basis (e_1, e_2) of $\mathbb{V}^2$, a parameterization of the circle with radius R and center $p \in \mathbb{E}^2$ is

$$c(t) = e_1(R\cos t) + e_2(R\sin t)$$

- Its velocity is

$$\dot{c}(t) = R(-e_1 \sin t + e_2 \cos t)$$

and its speed is

$$\sigma = |\dot{c}| = R$$

- The Frenet-Serret frame is

$$f_1(t) = \frac{\dot{c}(t)}{|\dot{c}(t)|} = -e_1 \sin t + e_2 \cos t$$

$$f_2(t) = -e_1 \cos t - e_2 \sin t$$

Frenet-Serret Equations for a Circle

- Differentiating the Frenet-Serret frame, we get

$$\begin{aligned}
\frac{1}{\sigma}\frac{d}{dt}\begin{bmatrix} f_1 & f_2 \end{bmatrix} &= \frac{1}{R}\frac{d}{dt}\begin{bmatrix} -e_1 \sin t + e_2 \cos t & -e_1 \cos t - e_2 \sin t \end{bmatrix} \\
&= \frac{1}{R}\begin{bmatrix} -e_1 \cos t - e_2 \sin t & e_1 \sin t - e_2 \cos t \end{bmatrix} \\
&= \frac{1}{R}\begin{bmatrix} f_2 & -f_1 \end{bmatrix} \\
&= \begin{bmatrix} f_1 & f_2 \end{bmatrix}\begin{bmatrix} 0 & -\frac{1}{R} \\ \frac{1}{R} & 0 \end{bmatrix}
\end{aligned}$$

- The curvature of a circle of radius R is $\kappa = \frac{1}{R}$

Example: Spiral

- Given $p \in \mathbb{E}^2$ and an orthonormal basis (e_1, e_2) of $\mathbb{V}^2$,

$$c(t) = e_1(t \cos t) + e_2(t \sin t)$$

  is a spiral
- Its velocity is

$$\dot{c}(t) = e_1(\cos t - t \sin t) + e_2(\sin t + t \cos t)$$

  and its speed is given by

$$\sigma^2 = (\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 = 1 + t^2$$

Frenet-Serret Frame of a Spiral

$$
\begin{aligned}
f_1(t) &= \frac{\dot{c}(t)}{|\dot{c}(t)|} \\
&= \frac{e_1(\cos t - t\sin t) + e_2(\sin t + t\cos t)}{\sqrt{1+t^2}} \\
f_2(t) &= \frac{-e_1(\sin t + t\cos t) + e_2(\cos t - t\sin t)}{\sqrt{1+t^2}}
\end{aligned}
$$

Curvature of a Spiral

- $$\dot{f}_1 = \frac{e_1(-2\sin t - t\cos t) + e_2(2\cos t - t\sin t)}{\sqrt{1+t^2}} - \frac{t}{(1+t^2)} f_1$$

- Therefore,

$$\begin{aligned}
\kappa &= \frac{1}{\sigma} f_2 \cdot \dot{f}_1 \\
&= \frac{(\sin t + t\cos t)(2\sin t + t\cos t)}{(1+t^2)^{3/2}} \\
&\quad + \frac{(\cos t - t\sin t)(2\cos t - t\sin t)}{(1+t^2)^{3/2}} \\
&= \frac{2+t^2}{(1+t^2)^{3/2}} \\
&= (1+t^2)^{-3/2} + (1+t^2)^{-1/2}
\end{aligned}$$