MATH-UA 377 Differential Geometry
Curves in Affine and Euclidean Space

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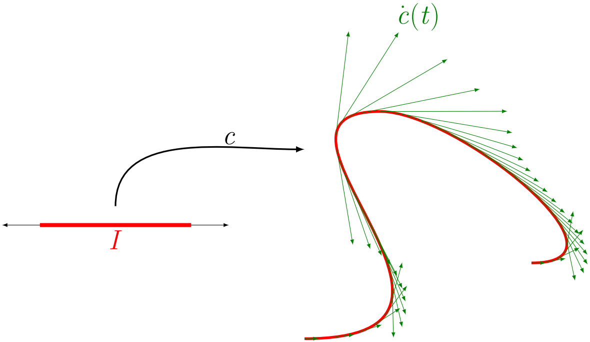
START RECORDING
LIVE TRANSCRIPTION

Geometric Properties of a Curve in Affine or Euclidean Space

- Use parameterized curves
- A geometric property of a curve should not depend on the parameterization
- An affine geometric property of a curve does not depend on affine transformations of the curve
- A Euclidean geometric property of a curve does not depend on isometries of the curve

Parameterized Curve in Affine Space



- A parameterized curve is a C^2 map $c : I \to \mathbb{A}^m$, where $I \subset \mathbb{R}$ is a connected nonempty interval
- The velocity of c is defined to be the derivative of c, $v = \dot{c} : I \to \mathbb{V}^m$, where

$$v(t) = \dot{c}(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h} \in \mathbb{V}^m$$

Reconstruct Curve From Velocity

- Consider a parameterized curve $c : [a, b] \to \mathbb{A}^m$ that starts at $c(a) = p$ and has velocity $\dot{c} = v$
- The Fundamental Theorem of Calculus says that if a function $f : [a, b] \to \mathbb{R}$ has a continuous derivative $\dot{f} : [a, b] \to \mathbb{R}$, then for any $t \in [a, b]$,

$$f(t) = f(a) + \int_{s=a}^{s=b} \dot{f}(s)\, ds$$

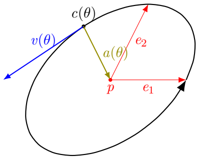
- The same holds for a map $c : [a, b] \to \mathbb{A}^m$ and therefore for any $t \in [a, b]$,

$$\begin{aligned} c(t) &= c(a) + \int_{s=a}^{s=t} \dot{c}(s)\, dx \\ &= p + \int_{s=a}^{s=t} v(s)\, ds \end{aligned}$$

Velocity and Acceleration

- If the velocity always points in the same direction, then the curve is straight
- The acceleration of c is defined to be the second derivative of c:
$$a(t) = \dot{v}(t) = \ddot{c}(t) = \lim_{h \to 0} \frac{\dot{c}(t+h) - \dot{c}(t)}{h}$$
- If the acceleration is always pointing in the same direction as the velocity, then the curve is straight
- If acceleration points in a different direction from velocity, then the curve turns

Parameterized Ellipse



- Given a point $p \in \mathbb{A}^2$ and a basis (e_1, e_2) of $\mathbb{V}^2$, consider the curve

$$c(\theta) = p + e_1 \cos\theta + e_2 \sin\theta,$$

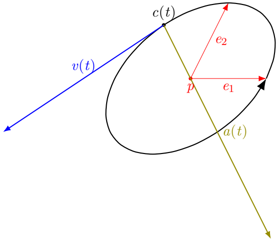
- The velocity is

$$v(\theta) = \dot{c}(\theta) = -e_1 \sin\theta + e_2 \cos\theta$$

- The acceleration is

$$a(\theta) = \ddot{c}(\theta) = -e_1 \cos\theta - e_2 \sin\theta = -(c(\theta) - p)$$

Parameterized Ellipse



- Given $p \in \mathbb{A}^2$ and a basis (e_1, e_2) of $\mathbb{V}^2$, let

$$c(t) = p + e_1 \cos 2t + e_2 \sin 2t$$

- The velocity is

$$v(t) = \dot{c}(t) = 2(-e_1 \sin 2t + e_2 \cos 2t)$$

- The acceleration is

$$a(t) = \ddot{c}(t) = -4(e_1 \cos 2t + e_2 \sin 2t) = -4(c(t) - p)$$

Parameterized Ellipse

- Given $p \in \mathbb{A}^2$ and a basis (e_1, e_2) of $\mathbb{V}^2$, let

$$c(t) = p + e_1 \cos t^2 + e_2 \sin t^2$$

- The velocity is

$$v(t) = \dot{c}(t) = 2t(-e_1 \sin t^2 + e_2 \cos t^2)$$

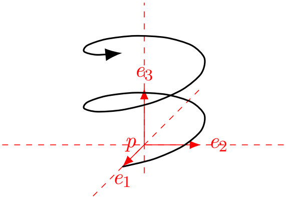
- The acceleration is

$$\begin{aligned} a(t) &= \ddot{c}(t) \\ &= 2(-e_1 \sin t^2 + e_2 \cos t^2) - 2t(e_1 \cos t^2 + e_2 \sin t^2) \\ &= 2u(t) - 2t(c(t) - p), \end{aligned}$$

where

$$u(t) = -e_1 \sin t^2 + e_2 \cos t^2$$

Helix in Affine 3-Space



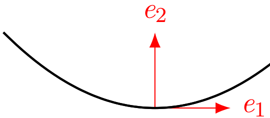
- Given $p \in \mathbb{A}^3$ and a basis (e_1, e_2, e_3) of $\mathbb{V}^3$, let

$$c(\theta) = e_1 \cos(\theta) + e_2 \sin(\theta) + e_3\,\theta,\ \ 0 \le \theta \le 2\pi$$

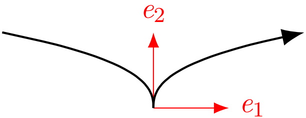
- The velocity is $v(\theta) = -e_1 \sin(\theta) + e_2 \cos(\theta) + e_3$
- The acceleration is $a(\theta) = -e_1 \cos(\theta) - e_2 \sin(\theta)$
- If $\mathbb{A}^3 = \mathbb{R}^3$, $p = (0, 0, 0)$, $e_1 = \langle 1, 0, 0\rangle$, $e_2 = \langle 0, 1, 0\rangle$, $e_3 = \langle 0, 0, 1\rangle$, then the formulas above become

$$\begin{aligned} c(\theta) &= (\cos\theta, \sin\theta, \theta) \\ v(\theta) &= \langle -\sin\theta, \cos\theta, 1\rangle \\ a(\theta) &= \langle -\cos\theta, -\sin\theta, 0\rangle \end{aligned}$$

Assume Nonzero Velocity



$$c(t) = e_1 2t + e_2 t^2$$
$$\dot{c}(t) = e_1 2 + e_2 2t$$
$$\dot{c}(0) = e_1 2 + e_2 0 \neq 0$$



$$c(t) = e_1 2t^5 + e_2 t^2$$
$$\dot{c}(t) = e_1 10t^4 + e_2 2t$$
$$\dot{c}(0) = e_1 0 + e_2 0$$

- If $v(t) = 0$ for some t, then the shape of the curve at $c(t)$ can have a kink
- We will always assume that a parameterized curve $c : I \to \mathbb{A}^m$ is C^2 (so acceleration can be defined) and, for every $t \in I$, $\dot{c}(t) \neq 0$

Parameterized Curve in Euclidean Space

- A parameterized curve in $\mathbb{E}^m$ is a C^2 map $c : I \to \mathbb{E}^m$ such that $\dot{c}(t) \neq 0$, for every $t \in I$
- The speed $\sigma : I \to [0, \infty)$ is defined to be the magnitude of velocity,

$$\sigma(t) = |v(t)| = |\dot{c}(t)|$$

- For each $t \in I$, since $v(t) \neq 0$, the speed is always positive,

$$\sigma(t) = |v(t)| > 0$$

- Since speed is the derivative of distance with respect to time, define the length of a curve $c : [t_0, t_1] \to \mathbb{E}^m$ to be

$$\ell = \int_{t=t_0}^{t=t_1} \sigma(t)\, dt = \int_{t=t_0}^{t=t_1} |\dot{c}(t)|\, dt$$

Speed and Arclength Functions

- Given a curve $c : [a, b] \to \mathbb{E}^m$ and $t \in [a, b]$, the arclength function, relative to t_0, is defined to be $s : [a, b] \to \mathbb{R}$, where

$$s(t) = \int_{\tau=t_0}^{\tau=t} |\dot{c}(\tau)| \, d\tau$$
$$= \int_{\tau=t_0}^{\tau=t} \sigma(\tau) \, d\tau$$

- $s(t) = \begin{cases} \text{distance from } s(t_0) \text{ to } s(t) \text{ if } t \geq t_0 \\ -(\text{distance from } s(t_0) \text{ to } s(t)) \text{ if } t \geq t_0 \end{cases}$
- Since $\sigma > 0$, the arclength function is strictly increasing

Acceleration

- The velocity of a curve $c : I \to \mathbb{E}^m$ can be written as

$$v = \dot{c} = \sigma u,$$

where σ is the speed and

$$u = \frac{\dot{c}}{|\dot{c}|}$$

is a unit vector giving the direction of the curve at each point

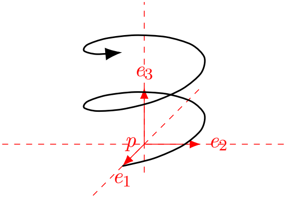
- Therefore, the acceleration is given by

$$a = \dot{v} = \dot{\sigma} u + \sigma \dot{u}$$

- $\dot{\sigma}$ measures the rate of change of speed, which is unrelated to the shape of the curve and therefore not geometric
- $\dot{u}$ measures the rate of change of direction, which contains geometric information but also depends on the speed
- Observe that, since $|u|^2 = 1$,

$$0 = \frac{d}{dt}(u \cdot u) = 2u \cdot \dot{u} \implies u \perp \dot{u}$$

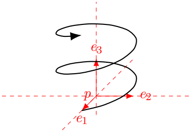
Example: Helix in Euclidean 3-Space



- Given $p \in \mathbb{E}^3$ and an orthonormal basis (e_1, e_2, e_3) of $\mathbb{V}^3$,

$$\begin{aligned}
c(\theta) &= e_1 \cos(\theta) + e_2 \sin(\theta) + e_3\,\theta \\
v(\theta) &= -e_1 \sin(\theta) + e_2 \cos(\theta) + e_3 \\
\sigma(\theta) &= \sqrt{2} \\
u(\theta) &= \frac{1}{\sqrt{2}}(-e_1 \sin(\theta) + e_2 \cos(\theta) + e_3) \\
\dot{u}(\theta) &= -\frac{1}{\sqrt{2}}(e_1 \cos(\theta) + e_2 \sin(\theta)) \\
a(\theta) &= -e_1 \cos(\theta) - e_2 \sin(\theta) \\
&= \dot{\sigma}(\theta) u(\theta) + \sigma(\theta) \dot{u}(\theta)
\end{aligned}$$

Example: Same Helix With Different Parameterization



$$
\begin{aligned}
\tilde{c}(t) &= e_1 \cos(t + 3t^2) + e_2 \sin(t + 3t^2) + e_3(t + 3t^2) \\
\tilde{v}(t) &= (1 + 6t)(-e_1 \sin(t + 3t^2) + e_2 \cos(t + 3t^2) + e_3) \\
\tilde{\sigma}(t) &= (1 + 6t)\sqrt{2} \\
\dot{\tilde{\sigma}}(t) &= 6\sqrt{2} \\
\tilde{u}(t) &= \frac{1}{\sqrt{2}}(-e_1 \sin(t + 3t^2) + e_2 \cos(t + 3t^2) + e_3) \\
\tilde{u}'(t) &= -(1 + 6t)\frac{1}{\sqrt{2}}(e_1 \cos(t + 3t^2) + e_2 \sin(t + 3t^2)) \\
\tilde{a}(t) &= 6(-e_1 \sin(t + 3t^2) + e_2 \cos(t + 3t^2) + e_3) \\
&\quad - (1 + 6t)^2(e_1 \cos(t + 3t^2) + e_2 \sin(t + 3t^2)) \\
&= \dot{\tilde{\sigma}}(t)\tilde{u}(t) + \tilde{\sigma}(t)\dot{\tilde{u}}(t)
\end{aligned}
$$

Speed is not Geometric But Arclength is

- A curve has many different parameterizations
- The shape of a curve does not depend on the speed of a parameterization
- A geometrically invariant parameterization is by arclength
- Given $0 \leq s \leq \ell$, let $\hat{c}(s) \in \mathbb{E}^m$ be the point on the curve whose distance along the curve from the start of the curve is equal to s
- If $\hat{c}$ is differentiable, then this means that

$$\int_{\tau=0}^{\tau=s} |\hat{\sigma}(s)| \, ds = s,$$

- Differentiating this, we get

$$\hat{\sigma}(s) = 1$$

- Arclength parameterization is also called unit speed parameterization

Example: Arclength Parameterization of Helix

- Helix in $\mathbb{E}^3$

$$c(\theta) = e_1 \cos(\theta) + e_2 \sin(\theta) + e_3 \,\theta$$
$$v(\theta) = -e_1 \sin(\theta) + e_2 \cos(\theta) + e_3$$
$$\sigma(\theta) = \sqrt{2}$$

- Suppose $s(\theta)$ is the arclength of curve from $c(0)$ to $c(\theta)$
- Since $\dot{s}(\theta) = \sigma(\theta) = \sqrt{2}$ and $s(0) = 0$,

$$s(\theta) = \sqrt{2}\theta$$

- We want $\hat{c}(s(\theta)) = c(\theta)$ and therefore

$$\hat{c}(s) = c\left(\frac{s}{\sqrt{2}}\right) = e_1 \cos\left(\frac{s}{\sqrt{2}}\right) + e_2 \sin\left(\frac{s}{\sqrt{2}}\right) + e_3 \,\frac{s}{\sqrt{2}}$$

Shape of a Curve in Euclidean Space

- The arclength parameterization is a unique parameterization defined purely in terms of the geometric structure of the curve
- Acceleration measures how curved the curve is
- With respect to a unit speed parameterization,

$$\begin{aligned} v(s) &= \sigma u \\ &= u \\ a &= \dot{v} \\ &= \dot{u} \end{aligned}$$

- Recall that $\dot{u} \cdot u = 0$
- Define the curvature function $\kappa : I \to [0, \infty)$ to be

$$\kappa(s) = |\dot{u}(s)| = |a(s)|$$

Example: Curvature of Helix

- Unit speed parameterization of helix in $\mathbb{E}^3$

$$c(s) = e_1 \cos\left(\frac{s}{\sqrt{2}}\right) + e_2 \sin\left(\frac{s}{\sqrt{2}}\right) + e_3 \frac{s}{\sqrt{2}}$$

$$v(s) = \frac{1}{\sqrt{2}}\left(-e_1 \sin\left(\frac{s}{\sqrt{2}}\right) + e_2 \cos\left(\frac{s}{\sqrt{2}}\right) + e_3\right)$$

$$a(s) = \frac{1}{2}\left(e_1 \cos\left(\frac{s}{\sqrt{2}}\right) + e_2 \sin\left(\frac{s}{\sqrt{2}}\right)\right)$$

- The curvature of this helix is

$$\kappa(s) = |a(s)| = \frac{1}{2}$$