MATH-UA 377 Differential Geometry
Euclidean Vector Space
Orthonormal Basis
Orthogonal Transformations
Orthogonal Matrices

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START RECORDING
LIVE TRANSCRIPTION

Euclidean Vector Space

- A Euclidean vector space consists of
  - A vector space $\mathbb{V}$
  - A positive definite symmetric bilinear function

$$\mathbb{V} \times \mathbb{V} \to \widehat{R}$$
$$(v_1, v_2) \mapsto v_1 \cdot v_2$$

  - Usually called inner product, but we say dot product
  - No explicit formula for the dot product is provided or needed
- An abstract vector space itself does not come automatically with an inner product
- There are infinitely many possible inner products on a vector space

Important Special Values of Dot Product

- $v \cdot w = 0$ iff they are orthogonal, normal, perpendicular
- $v \cdot w = |v||w|$ iff there is a scalar $r \geq 0$ such that

$$\text{either } v = rw \text{ or } w = rv$$

- $v \cdot w = -|v||w|$ iff there is a scalar $r \leq 0$ such that

$$\text{either } v = rw \text{ or } w = rv$$

Dot Product With Respect to Basis

- Suppose $\mathbb{V}^m$ is an abstract vector space with a dot product
- Suppose $E = (e_1, \dots, e_m)$ is a basis of $\mathbb{V}^m$
- The dot product of any two vectors

$$v = a^1 e_1 + \cdots + a^m e_m = Ea$$
$$w = b^1 e_1 + \cdots + b^m e_m = Eb$$

is

$$\begin{aligned} v \cdot w &= (a^1 e_1 + \cdots + a^m e_m) \cdot (b^1 e_1 + \cdots b^m e_m) \\ &= a^1 b^1 e_1(\cdot e_1) + \cdots + a^m b^1 (e_m \cdot e_1) \\ &\quad + \cdots + a^1 b^m e_1 \cdot e_m + \cdots a^m b^m e_m \cdot e_m \\ &= \begin{bmatrix} a^1 & \cdots & a^m \end{bmatrix} \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ \vdots & & \vdots \\ G_{m1} & \cdots & G_{mm} \end{bmatrix} \begin{bmatrix} b^1 \\ \vdots \\ b^m \end{bmatrix} \\ &= a^t G b, \end{aligned}$$

where $G_{ij} = G_{ji} = e_i \cdot e_j$

Dot Product With Respect to Basis

- If $E = (e_1, \ldots, e_m)$ is a basis of $\mathbb{V}$, then a dot product on $\mathbb{V}$ is uniquely determined by the symmetric matrix

$$G = E^t E = \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ \vdots & & \vdots \\ G_{m1} & \cdots & G_{mm} \end{bmatrix},$$

  where $G_{ij} = e_i \cdot e_j$
- Observe that both indices of G are subscripts
- G is a positive definite symmetric 2-tensor written with respect to the basis E

Symmetric Bilinear Functions on $\widehat{R}^2$

- On $\widehat{R}^2$, if
$$Q(\langle a^1, a^2\rangle, \langle b^1, b^2\rangle) = 2a^1b^1 + 3a^2b^2 = \begin{bmatrix} a^1 & a^2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$
then
$$Q(a, a) = 2(a^1)^2 + 3(a^2)^2 \geq 0, \text{ and } Q(a, a) = 0 \implies a = 0$$
- If
$$Q(\langle a^1, a^2\rangle, \langle b^1, b^2\rangle) = a^1b^1 = \begin{bmatrix} a^1 & a^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$
then
$$Q(a, a) = (a^1)^2 \geq 0, \text{ but } Q(\langle 0, 1\rangle, \langle 0, 1\rangle) = 0$$
- If
$$Q(\langle a^1, a^2\rangle, \langle b^1, b^2\rangle) = a^1b^2 + a^2b^1 = \begin{bmatrix} a^1 & a^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$
then
$$Q(\langle 1, -1\rangle, \langle 1, -1\rangle) = -2 < 0$$

Positive Definite Symmetric Bilinear Functions on $\widehat{R}^2$

- A symmetric bilinear function

$$Q(\langle a^1, a^2 \rangle, \langle b^1, b^2 \rangle) = \begin{bmatrix} a^1 & a^2 \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$

is positive definite iff the following hold:

$$\begin{aligned} G_{11} &> 0 \\ G_{22} &> 0 \\ \det G = G_{11} G_{22} - (G_{12})^2 &> 0 \end{aligned}$$

Important Special Case of Dot Product

- Suppose $\mathbb{V}$ is a Euclidean vector space
- Suppose $L \subset \mathbb{V}$ is a linear subspace
- The dot product on $\mathbb{V}$ can be restricted to L
- The dot product restricted on L is positive definite

Orthonormal Set

- A set S of vectors is orthonormal with respect to a dot product, if for any two vectors $v, w \in S$,

$$v \cdot w = \begin{cases} 1 & \text{if } v = w \\ 0 & = \text{ if } v \neq w \end{cases}$$

- The vectors in an orthonormal set are linearly independent
  - Suppose
    $$a^1 v_1 + \cdots + a^k v_k = 0,$$
    where $v_1, \ldots, v_k \in S$ and $a^1, \ldots, a^k \in \mathbb{R}$
  - It follows that for each $1 \leq j \leq k$,
    $$\begin{aligned} 0 &= v_j \cdot (a^1 v_1 + \cdots + a^k v_k) \\ &= a_1(v_j \cdot v_1) + \cdots + a^k(v_j \cdot v_k) \\ &= a^j \end{aligned}$$
  - It follows that the number of elements in S is at most the dimension of the vector space

Orthonormal Basis

- If $\mathbb{V}$ is Euclidean vector space, then a basis $(e_1, \ldots, e_m)$ is orthonormal if
$$e_i \cdot e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
- Equivalently the matrix G is equal to the identity matrix
$$G = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$
- If $(e_1, \ldots, e_m)$ is orthonormal and $v = v^1 e_1 + \cdots + v^m e_m$, then
$$v^i = v \cdot e_i$$
- Fundamental example: Standard basis of $\widehat{R}^m$, for each $1 \leq i \leq m$,
$$e_i = \langle 0, \ldots, 1, \ldots \rangle$$
with a 1 in the i-th slot and 0s elsewhere

Constructing an Orthonormal Basis

- If a Euclidean vector space $\mathbb{V}$ has dimension m, then it has a at least one (not necessarily orthonormal) basis $F = (f_1, \ldots, f_m)$.
- Set $e_1 = \frac{f_1}{|f_1|}$
- Set $h_1 = f_2 - (f_2 \cdot e_1)e_1$
- h_1 is nonzero and orthogonal to e_1
- Set $e_2 = \frac{h_1}{|h_1|}$
- Proceed by induction: Assume that $e_1, \ldots, e_k$ are orthonormal, where $1 \leq k < m$
- Set $h_{k+1} = f_{k+1} - (f_{k+1} \cdot e_1)e_1 - \cdots - (f_{k+1} \cdot e_k)e_k$
- h_{k+1} is nonzero and orthogonal to $e_1, \ldots, e_k$
- Set $e_{k+1} = \frac{h_{k+1}}{|h_{k+1}|}$
- $e_1, \ldots, e_{k+1}$ are orthonormal

Orthogonal Maps

- If $\mathbb{V}^m$ and $\mathbb{W}^n$ are Euclidean vector spaces, a map

$$F : \mathbb{V}^m \to \mathbb{W}^n$$

  is called *orthogonal*, if for any $v_1, v_2 \in \mathbb{V}$,

$$(F(v_1)) \cdot (F(v_2)) = v_1 \cdot v_2$$

- Equivalently, F is orthogonal if it preserves the magnitude of a vector and the angle between two vectors

An Orthogonal Map is Linear

- If a map $F : \mathbb{V}^m \to \mathbb{W}^n$ is orthogonal, then for any orthonormal basis $(e_1, \ldots, e_m)$, the ordered set

$$(f_1, \ldots, f_m) = (F(e_1), \ldots, F(e_m))$$

is an orthonormal set, because

$$F(e_i) \cdot F(e_j) = e_i \cdot e_j = \delta_{ij}$$

- Consequence: $m \leq n$
- F is linear, because for any $v = a^1 e_1 + \cdots + a^m e_m$,

$$\begin{aligned} f_i \cdot F(v) &= F(e_i) \cdot F(a^1 e_1 + \cdots + a^m e_m) \\ &= e_i \cdot (a^1 e_1 + \cdots a^m e_m) \\ &= a^i \\ \implies F(v) &= a^1 f_1 + \cdots a^m f_m \\ &= a^1 F(e_1) + \cdots + a^m F(e_m) \end{aligned}$$

Orthogonal matrices

- An orthogonal map $F : \mathbb{V}^m \to \mathbb{V}^m$ is called an orthogonal transformation
- If $E = (e_1, \dots, e_m)$ is an orthornomal basis, then so is
$$F = (f_1, \dots, f_m) = (F(e_1), \dots, F(e_m))$$
- There is an invertible square matrix M such that
$$F = EM$$
- Observe that if $M_k \in \widehat{R}^m$ is the k-th column of M,
$$\begin{aligned} f_i \cdot f_j &= (M_i^1 e_1 + \cdots + M_i^m e_m) \cdot (M_j^1 e_1 + \cdots + M_j^m e_m) \\ &= M_i^1 M_j^1 + \cdots + M_i^m M_j^m \\ &= M_i \cdot M_j, \end{aligned}$$
- It follows that L is an orthogonal transformation if and only if the columns $M_1, \dots, M_m$ form an orthonormal basis of $\widehat{R}^m$
- This is equivalent to $M^T M = I$
- Any such matrix is called an orthogonal matrix
- Let $O(m)$ be the set of all orthogonal matrices

Orthogonal Transformations of $\widehat{R}^2$

- A linear transformation $T : \widehat{R}^2 \to \widehat{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

  is orthogonal if

$$a^2 + c^2 = 1$$
$$b^2 + d^2 = 1$$
$$ac + bd = 0$$

- This holds if and only if there is an angle θ such that

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or } \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- The matrix on the left is rotation counterclockwise by angle θ
- The matrix on the right is rotation counterclockwise by angle θ followed by a reflection about the y-axis

3-Dimensional Orthogonal Transformations

If $T : \mathbb{V}^3 \to \mathbb{V}^3$ is an orthogonal transformation, then

- There is a line ℓ through the origin (i.e., a 1-dimensional linear subspace) such that $T(\ell) = \ell$ and for any $v \in \ell$, $T(v) = \pm v$
- If $T(v) = v$ for any $v \in \ell$, then T is a rotation around ℓ