MATH-UA 377 Differential Geometry
EucIdean Vector Space
Euclidean Space

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START RECORDING
LIVE TRANSCRIPTION

Geometry of 3-Dimensional Abstract Vector and Affine Spaces

- Abstract vector space
  - Arrows, lines and planes through the origin
  - Can measure relative lengths of parallel vectors
  - No consistent way to compare the length of two vectors pointing in different directions
  - No way to measure the angle between two vectors
- Abstract affine space
  - Points, lines, planes
  - Can measure distance between two points relative to the distance between two other points only if all lie on a line
  - No consistent way to compare the distance between two pairs of points that lie on two non-parallel lines
  - No way to the measure the angle between two intersecting lines
- Measurement of lengths, distances, angles require something more

Euclidean Geometry in $\mathbb{R}^m$

- Classical 2 and 3 dimensional Euclidean geometry starts with axioms (definitions and assumptions) and deduces theorems from them rigorous logic
- Cartesian geometry starts with $\widetilde{\mathbb{R}}^m$ and deduces the same theorems from algebraic calculations using the vector space properties and the dot product on $\mathbb{R}^m$, $m = 2, 3$
- In either approach no pictures are needed to prove the theorems
- Pictures, however, provide a useful guide

Dot Product on $\mathbb{R}^m$

- Recall definition of dot product: If $v = \langle v^1, \ldots, v^m \rangle$ and $w = \langle w_1, \ldots, w_m \rangle$, then

$$v \cdot w = v^1 w^1 + \cdots + v^m w^m = v^T w,$$

where

$$v^T = \begin{bmatrix} v^1 \cdots v^m \end{bmatrix}$$

- The length of a vector v is $|v|$, where

$$|v|^2 = v \cdot v$$

- The angle between vectors v_1 and v_2 is θ, where

$$v_1 \cdot v_2 = |v_1||v_2| \cos \theta$$

Observe that this uniquely determines an angle $0 \leq \theta \leq \pi$

Key Properties of the Dot Product

- Bilinearity

$$(v_1 + v_2) \cdot w = v_1 \cdot w + v_2 \cdot w$$
$$(rv) \cdot w = r(v \cdot w)$$

- Symmetry

$$v \cdot w = w \cdot v$$

- Positive definiteness

$$v \cdot v \geq 0$$

and

$$v \cdot v = 0 \iff v = 0$$

Cauchy-Schwarz Inequality

- If $|v| = |w| = 1$, then $-1 \leq v \cdot w \leq 1$
- Proof

$$
\begin{aligned}
0 &\leq (v-w)\cdot(v-w) && \text{(positive definiteness)}\\
&= v\cdot(v-w) - w\cdot(v-w) && \text{(bilinearity)}\\
&= v\cdot v - v\cdot w - w\cdot v + w\cdot w && \text{(bilinearity)}\\
&= |v|^2 + |w|^2 - v\cdot w - w\cdot v && \text{(definition of norm)}\\
&= |v|^2 + |w|^2 - 2v\cdot w && \text{(commutativity)}\\
&= 2(1 - v\cdot w) && \text{(assumption)}
\end{aligned}
$$

$$\implies \quad v\cdot w \leq 1$$

- Since $|-v| = 1$, the above inequality also implies that

$$(-v)\cdot w \leq 1 \implies -1 \leq v\cdot w$$

- If $v\cdot w = 1$, then $(v-w)\cdot(v-w) = 0$ and therefore $v = w$

General Form of Cauchy-Schwarz Inequality

- For any $v, w \in \mathbb{V}$,
$$v \cdot w \leq |v||w|$$
and equality holds if and only if one of them is a nonnegative scalar multiple of the other
- Proof:
  - If either v or w is 0, then equality holds
  - If both are nonzero, then

$$\begin{aligned}
\left(\frac{v}{|v|}\right) \cdot \left(\frac{v}{|v|}\right) &= \frac{1}{|v|^2} v \cdot v && \text{(bilinearity)} \\
&= \frac{1}{|v|^2} |v|^2 && \text{definition of norm)} \\
&= 1
\end{aligned}$$

Therefore,

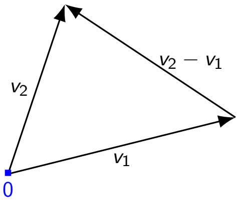
$$\left(\frac{v}{|v|}\right) \cdot \left(\frac{w}{|w|}\right) \leq 1 \iff v \cdot w \leq |v||w| \qquad \text{(bilinearity)}$$

Equality Case of Cauchy-Schwarz

- If $v, w \neq 0$, then $v \cdot w = |v||w| \iff w$ points in same direction as v
- Proof:

$$\begin{aligned}
v \cdot w &= |v||w| \\
\implies \left(\frac{v}{|v|}\right) \cdot \left(\frac{w}{|w|}\right) &= 1 && \text{(bilinearity)} \\
\implies \frac{v}{|v|} &= \frac{w}{|w|} && \text{(Equality case)} \\
\implies w &= \frac{|w|}{|v|} v && \text{(scalar multiplication)}
\end{aligned}$$

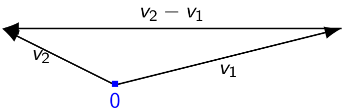
Triangle Inequality



- $|v_2 - v_1| \leq |v_1| + |v_2|$
- Proof:

$$
\begin{aligned}
|v_2 - v_1|^2 &= (v_2 - v_1) \cdot (v_2 - v_1) && \text{(definition of norm)} \\
&= v_1 \cdot v_1 + v_2 \cdot v_2 - 2 v_1 \cdot v_2 && \text{(properties of dot product)} \\
&= |v_1|^2 + |v_2|^2 - 2 v_1 \cdot v_2 && \text{(definition of norm)} \\
&\leq |v_1|^2 + |v_2|^2 + 2 |v_1||v_2| && \text{(Cauchy-Schwarz)} \\
&= (|v_1| + |v_2|)^2 && \text{(algebra)}
\end{aligned}
$$

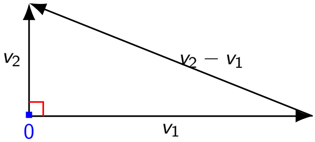
Equality Case of Triangle inequality



- If $v_1, v_2 \neq 0$, then $|v_2 - v_1| = |v_1| + |v_2| \iff$ the vectors point in opposite directions
- Proof:

$$
\begin{aligned}
|v_2 - v_1|^2 &= (|v_1| + |v_2|)^2 \\
|v_1|^2 + |v_2|^2 - 2v_1 \cdot v_2 &= |v_1|^2 + |v_2|^2 + 2|v_1||v_2| \\
-v_1 \cdot v_2 &= |v_1||v_2| \\
(-v_1) \cdot v_2 &= |-v_1||v_2| \\
-v_1 & \text{ points in same direction as } v_2 \\
v_1 & \text{ points in opposite direction to } v_2
\end{aligned}
$$

Pythagorean Theorem



- If $v_1 \cdot v_2 = 0$, then

$$|v_2 - v_1|^2 = |v_1|^2 + |v_2|^2$$

- Proof:

$$\begin{aligned} |v_2 - v_1|^2 &= (v_2 - v_1) \cdot (v_2 - v_1) \\ &= v_2 \cdot v_2 - v_1 \cdot v_2 - v_2 \cdot v_1 + v_1 \cdot v_1 \\ &= |v_1|^2 + |v_2|^2 \end{aligned}$$

Proofs Used Only Properties of the Dot Product

- Formula for the dot product never used
- Geometric arguments never used
- Proof used only the following:
  - Properties of dot product
  - Algebra
- Therefore, the Cauchy-Schwarz inequality, triangle inequality, and Pythagorean Theorem hold for any function that has the same properties as the dot product

Positive definite symmetric bilinear function

- Consider a function $Q : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$
- Q is bilinear if the following hold:

$$
\begin{aligned}
Q(v_1 + v_2, w) &= Q(v_1, w) + Q(v_2, w), \ \forall \ v_1, v_2, w \in \mathbb{V} \\
Q(v, w_1 + w_2) &= Q(v, w_1) + Q(v, w_2, \ \forall v, w_1, w_2 \in \mathbb{V} \\
Q(av, w) &= aQ(v, w), \ \forall v, w \in \mathbb{V}, \ \ a \in \mathbb{R} \\
Q(v, aw) &= aQ(v, w), \ \forall v, w \in \mathbb{V}, \ \ a \in \mathbb{R}
\end{aligned}
$$

- Q is symmetric, if

$$Q(v_1, v_2) = Q(v_2, v_1), \ \forall \ v_1, v_2 \in \mathbb{V}$$

- Q is positive definite, if for any $v \in \mathbb{V}$,

$$Q(v, v) \geq 0$$

  and

$$Q(v, v) = 0 \quad \Longleftrightarrow \quad v = 0$$

- If $\mathbb{V} = \mathbb{R}^m$ and $Q(v, w) = v \cdot w$, then Q is positive definite, symmetric, and bilinear

Cauchy-Schwarz inequality

- If $Q(v, v) = Q(w, w) = 1$, then
$$Q(v, w) \leq 1$$
with equality holding if and only if $v = w$
- Proof:
$$\begin{aligned} 0 &\leq Q(v - w, v - w) \\ &= Q(v, v) - Q(v, w) - Q(w, v) + Q(w, w) \\ &= 2 - 2Q(v, w) \\ &= 2(1 - Q(v, w)) \\ \implies \quad Q(v, w) &\leq 1 \end{aligned}$$
- Equality case follows by replacing inequalities by equalities
- Corollary:
$$-\sqrt{Q(v, v)Q(w, w)} \leq Q(v, w) \leq \sqrt{Q(v, v)Q(w, w)}$$

Inequalities satisfied by a positive definite symmetric bilinear function

- Suppose $Q : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is a positive definite symmetric bilinear function
- Cauchy-Schwarz inequality

$$Q(v, w) \leq \sqrt{Q(v, v) Q(w, w)}$$

  If equality holds, then there exists a nonnegative scalar r such that either $v = rw$ or $w = rv$
- Triangle inequality

$$\sqrt{Q(v + w, v + w)} \leq \sqrt{Q(v, v)} + \sqrt{Q(w, w)}$$

- If equality holds, then there exists a nonnegative scalar r such that either $v = rw$ or $w = rv$

Pythagorean Theorem

- If $Q(v, w) = 0$, we say that v is orthogonal to w
- If v is orthogonal to w, then

$$Q(v, v) + Q(w, w) = Q(v + w, v + w)$$

Example: Spce of Continuous Functions on $[0, 1]$

- Let $C([0,1])$ be the space of continuous functions with domain $[0,1]$
- $C([0,1])$ is a vector space, because
  - f and g are continous functions $\implies$ $f+g$ is continuous
  - f is a continuous function and $r \in \mathbb{R}$ $\implies$ rf is continuous
- Given two functions $f, g \in \mathbb{V}$, define

$$Q(f,g) = \int_{x=0}^{x=1} f(x)g(x)\,dx$$

- Q is a positive definite symmetric bilinear function

Integral Inequalities for Continuous Functions on $[0, 1]$

- Let f and g be continuous function with domain $[0, 1]$
- Cauchy-Schwarz inequality:

$$\int_{x=0}^{x=1} f(x)g(x)\,dx \le \left(\int_{x=0}^{x=1} (f(x))^2\,dx\right)^{1/2} \left(\int_{x=0}^{x=1} (g(x))^2\,dx\right)^{1/2}$$

  Equality holds if and only if $g = cf$ or $f = cg$
- Triangle inequality:

$$\left(\int_{x=0}^{x=1} (f(x)+g(x))^2\,dx\right)^{1/2} \le \left(\int_{x=0}^{x=1} (f(x))^2\,dx\right)^{1/2} + \left(\int_{x=0}^{x=1} (g(x))^2\,dx\right)^{1/2}$$

Inner Product Space

- An inner product space consists of a vector space $\mathbb{V}$ and a positive definite symmetric bilinear function $Q : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$
- Q is a function of two vectors and is called an inner product

Examples of Inner Product Spaces

- $\mathbb{V} = \mathbb{R}^2$ and

$$Q(\langle v^1, v^2\rangle, \langle w^1, w^2\rangle) = v^1 w^1 + v^2 w^2$$

- $\mathbb{V} = \mathbb{R}^2$ and

$$Q(\langle v^1, v^2\rangle, \langle w^1, w^2\rangle) = 2v^1 w^1 + 3v^2 w^2$$

- $\mathbb{V} = C([0,1])$ and

$$Q(f, g) = \int_{x=0}^{x=1} f(x) g(x)\, dx$$

- Example of a symmetric bilinear form that is not positive definite: $\mathbb{V} = \mathbb{R}^2$ and

$$Q(\langle v^1, v^2\rangle, \langle w^1, w^2\rangle) = v^1 w^1 - v^2 w^2$$

Euclidean Vector Space

- A Euclidean vector space is a finite dimensional inner product space
- For convenience, we will write the inner product as a dot product: Given $v, w \in \mathbb{V}$,

$$v \cdot w = Q(v, w)$$

- Define the length of a vector v to be $|v| \geq 0$, where

$$|v|^2 = v \cdot v$$

- The Cauchy-Schwarz inequality holds and therefore, if v and w are nonzero vectors in $\mathbb{V}$,

$$-1 \leq \frac{v \cdot w}{|v||w|} \leq 1$$

- We can therefore define the angle between v and w to be

$$\theta = \arccos \frac{v \cdot w}{|v||w|}$$

Euclidean space

- Euclidean space is an affine space whose tangent space is a Euclidean vector space
- If $\mathbb{E}$ is a Euclidean space with tangent space $\dot{\mathbb{E}}$, then we define the distance between two points $p, q \in \mathbb{E}$ to be

$$d(p, q) = |q - p| = \sqrt{(q - p) \cdot (q - p)}$$

- Triangle inequality: If $p, q, r \in \mathbb{E}$, then

$$d(p, r) \leq d(p, q) + d(q, r)$$

  with equality holding if and only if p, q, r are collinear and q lies between p and r

Differential geometry

- Classical Euclidean geometry
  - Euclidean space is the ambient space
    - Euclidean 2-space is the plane
    - Euclidean 3-space is our universe
  - Lines, planes, triangles, rectangles, parallelograms
  - Polygons, polytopes
  - Circles, ellipses, quadratic curves, quadric surfaces
  - Key tool: Geometric axioms
- Classical differential geometry
  - Curves and surfaces in ambient Euclidean space
  - Submanifolds in Euclidean m-space
  - Key tool: Differential and integral calculus
- Modern differential geometry
  - Geometric space is ambient space itself
  - Universe is curved
  - Introduced and studied by Riemann
  - Basis for Einstein's theory of general relativity