MATH-UA 377 Differential Geometry
Linear Functions and Maps
Affine Maps

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START RECORDING

Linear Functions and Maps

- Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces
- A function $f : \mathbb{V} \to \mathbb{R}$ is linear, if for any vectors $v, v_1, v_2 \in \mathbb{V}$ and scalar $r \in \mathbb{R}$,

$$f(v_1 + v_2) = f(v_1) + f(v_2)$$
$$f(rv) = rf(v)$$

- A map $L : \mathbb{V} \to \mathbb{W}$ is linear, if for any vectors $v, v_1, v_2 \in \mathbb{V}$ and scalar $r \in \mathbb{R}$,

$$L(v_1 + v_2) = L(v_1) + L(v_2)$$
$$L(rv) = rL(v)$$

Linear Functions and Maps on $\widehat{\mathbb{R}}^m$

- Any linear function on $\widehat{\mathbb{R}}^m$ is of the form

$$\ell(\langle v^1, \dots, v^m \rangle) = a_1 v^1 + \cdots + a_m v^m = \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix} \begin{bmatrix} v^1 \\ \cdots \\ v^m \end{bmatrix}$$

  and therefore, if vectors are column matrices, the linear functions are row matrices

- Any linear map

$$M : \widehat{\mathbb{R}}^m \to \widehat{\mathbb{R}}^n$$

  is given by an n-by-m matrix,

$$Mv = \begin{bmatrix} M_1^1 & \cdots & M_m^1 \\ \vdots & & \vdots \\ M_1^n & \cdots & M_m^n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix}$$

Differentiation is Linear

- Suppose P is the space of polynomials in a single variable x. The differentiation map

$$\begin{aligned} D : P &\to P \\ p(x) &\mapsto p'(x) \end{aligned}$$

  is a linear map
- The value of the derivative at the origin,

$$\begin{aligned} D_0 : P &\to \mathbb{R} \\ p(x) &\mapsto p'(0) \end{aligned}$$

  is a linear function

Linear Map with Respect to Basis

- Suppose
  - $L : \mathbb{V} \to \mathbb{W}$ is a linear map
  - $E = (e_1, \ldots, e_m)$ is a basis of $\mathbb{V}$
  - $F = (f_1, \ldots, f_n)$ is a basis of $\mathbb{W}$
- For each e_k, there is a unique vector $b_k = \langle b_k^1, \ldots, b_k^n \rangle$ such that

$$L(e_k) = b_k^1 f_1 + \cdots b_k^n f_n$$

- This defines a matrix

$$B = \begin{bmatrix} b_1^1 & \cdots & b_m^1 \\ \vdots & & \vdots \\ b_1^n & \cdots & b_m^n \end{bmatrix}$$

Linear Map as Matrix

- Given any $v = a^1 e_1 + \cdots + a^m v_m$, suppose

$$L(v) = f_1 c^1 + \cdots f_n c^n = FC$$

- Then

$$\begin{aligned} FC &= L(v) \\ &= L(a^1 e_1 + \cdots + a^m e_m) \\ &= a^1 L(e_1) + \cdots a^m L(e_m) \\ &= a^1(b_1^1 f_1 + \cdots + b_1^n f_n) + \cdots + a^m(b_m^1 f_1 + \cdots + b_m^n f_n) \\ &= \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} \begin{bmatrix} b_1^1 & \cdots b_m^1 \\ \vdots & & \vdots \\ b_1^n & \cdots b_m^n \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} \\ &= FBA \end{aligned}$$

- Therefore, $C = BA$ and

$$L(v) = L(EA) = F(BA)$$

Space of Linear Maps is a Vector Space

- Given vector spaces $\mathbb{V}$ and $\mathbb{W}$,

$$\mathrm{Hom}(\mathbb{V}, \mathbb{W}) = \{\text{linear maps } L : \mathbb{V} \to \mathbb{W}\}$$

  is a vector space
- Given a vector space $\mathbb{V}$,

$$\mathrm{gl}(\mathbb{V}) = \mathrm{Hom}(\mathbb{V}, \mathbb{V}) = \{\text{linear maps } L : \mathbb{V} \to \mathbb{V}\}$$

  is a vector space but

$$\mathrm{GL}(\mathbb{V}) = \mathrm{Aut}(\mathbb{V}) = \{\text{invertible linear maps } L : \mathbb{V} \to \mathbb{V}\}$$

  is not
- $\mathrm{GL}(\mathbb{V})$ is a group, where group multiplication is composition of maps

Basis and Dimension of $\text{Hom}(\mathbb{V}, \mathbb{W})$

- Let $\dim \mathbb{V} = m$ and $E = (e_1, \ldots, e_m)$ be a basis of $\mathbb{V}$
- Let $\dim \mathbb{W} = n$ and $F = (f_1, \ldots, f_n)$ be a basis of $\mathbb{W}$
- For each $1 \leq j \leq m$ and $1 \leq p \leq n$, define the linear map

$$\begin{aligned} L_{jp} : \mathbb{V} &\mapsto \mathbb{W} \\ a^1 e_1 + \cdots a^m e_m &\mapsto a^j f_p \end{aligned}$$

- The set, in any order,

$$\{L_{11}, \cdots, L_{mn}\}$$

  is a basis
- Therefore,

$$\dim \text{Hom}(\mathbb{V}, \mathbb{W}) = mn$$

Dual Vector Space

- An important special case is the space of linear functions
- Given a vector space $\mathbb{V}$, define its dual to be the vector space of linear functions on $\mathbb{V}$,

$$\mathbb{V}^* = \{\ell : \mathbb{V} \to \mathbb{R} \ : \ \ell \text{ is linear}\}$$

- We call an element of $\mathbb{V}^*$ a **covector** or **dual vector** or **1-tensor**
- If $\ell \in \mathbb{V}^*$ is nonzero, then

$$\ell^{-1}(0) = \{v \in \mathbb{V} \ : \ \ell(v) = 0\}$$

  is a codimension 1 linear subspace of $\mathbb{V}$
- The level sets of ℓ are parallel to $\ell^{-1}(0)$

Dual Basis

- Let $E = (e_1, \dots, e_m)$ be a basis of $\mathbb{V}$
- For each $1 \le i \le m$, define the linear function

$$\begin{aligned} \ell^i &: \mathbb{V} \to \mathbb{R} \\ a^1 e_1 + \cdots + a^m e_m &\mapsto a^i \end{aligned}$$

- Then $E^* = (\ell^1, \dots, \ell^m)$ is a basis of $\mathbb{V}^*$
- E^* is called the dual basis to E
- We will write E^* as a column vector of covectors

$$E^* = \begin{bmatrix} \ell^1 \\ \vdots \\ \ell^m \end{bmatrix}$$

- If $\ell = a_1\ell^1 + \cdots + a_m\ell^m$, we can write

$$\ell = \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix} \begin{bmatrix} \ell^1 \\ \vdots \\ \ell^m \end{bmatrix}$$

Affine Map

- If $\mathbb{A}$ and $\mathbb{B}$ are affine space, a map
$$M : \mathbb{A} \to \mathbb{B}$$
is **affine** if there exists a linear map
$$dM : \mathbb{V} \to \mathbb{W}$$
such that for any $p \in \mathbb{A}$ and $v \in \mathbb{V}$,
$$M(p + v) = M(p) + dM(v)$$
- Equivalently, for any $p, q \in \mathbb{V}$,
$$M(q) - M(p) = dM(q - p)$$

Directional Derivatives of an Affine Map

- If M is an affine map, then its directional derivative at p in a direction v is

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} (M(p+tv) - M(p)) &= \left.\frac{d}{dt}\right|_{t=0} dM(tv) \\ &= \left.\frac{d}{dt}\right|_{t=0} t\, dM(v) \\ &= dM(v) \end{aligned}$$

- dM is therefore the differential or Jacobian of M at any point p

Affine Map

- Given a point $p \in \mathbb{A}$, define the map $I_p : \mathbb{V} \to \mathbb{A}$ by

$$I_p(v) = p + v$$

- The inverse to I_p is $I_p^{-1} : \mathbb{A} \to \mathbb{V}$, where

$$I_p^{-1}(q) = q - p$$

- If $\mathbb{B}$ is an affine space with tangent space $\mathbb{W}$ and

$$M : \mathbb{A} \to \mathbb{B} \text{ is an affine map,}$$

  then

$$dM_p = I_{M(p)}^{-1} \circ M \circ I_p$$

- Given an affine map $M : \mathbb{A} \to \mathbb{B}$, we have the following commuting diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{M} & \mathbb{B} \\ {\scriptstyle I_p}\uparrow & & \uparrow{\scriptstyle I_{M(p)}} \\ \mathbb{V} & \stackrel{dM_p}{\dashrightarrow} & \mathbb{W} \end{array}$$