MATH-UA 377 Differential Geometry
Linear Subspaces
Basis and Dimension
Change of Basis and Coordinates
Affine Space
Affine Basis
Geometry of Vector and Affine Spaces

Deane Yang

Courant Institute of Mathematical Sciences
New York University

January 27, 2022

START RECORDING

Linear Subspaces

- A subset $L \subset V$ is a **linear subspace**, if is is closed under the vector addition and scalar multiplication defined on V
- A linear subspace is itself a vector space
- Example:

  $$L = \{\langle x^1, \ldots, x^m \rangle \in \widehat{R}^m \; : \; a_1 x^1 + \cdots + a_m x^m = 0\}$$

  is a linear subspace
- Example:

  $$L = \{\langle x^1, \ldots, x^m \rangle \in \widehat{R}^m \; : \; a_1 x^1 + \cdots + a_m x^m = 1\}$$

  is NOT a linear subspace

Linearly Independent Set of Vectors

- A finite set of vectors, $\{v_1, \ldots, v_k\} \subset V$, is **linearly dependent**, if there exist scalars $a^1, \ldots, a^k$, not all zero, such that

$$a^1 v_1 + \cdots a^k v_k = 0 \implies a^1 = \cdots = a^k = 0$$

- A finite set of vectors, $\{v_1, \ldots, v_k\} \subset V$, is **linearly independent**, if they are not linearly dependent
- A finite set of vectors, $\{v_1, \ldots, v_k\} \subset V$, is **linearly independent**, if, for any scalars $a^1, \ldots, a^m$,

$$a^1 v_1 + \cdots a^k v_k = 0 \implies a^1 = \cdots = a^k = 0$$

- Examples
  - $\{\langle 1, 0, 0\rangle, \langle 1, 1, 0\rangle, \langle 0, 0, 1\rangle\} \subset \widehat{R}^3$ is linearly independent
  - $\{\langle 1, 0, 0\rangle, \langle 1, 0, -1\rangle, \langle 0, 0, 1\rangle \subset \widehat{R}^3$ is linearly dependent

Span of a Set of Vectors

- Given a subset $S \subset V$, define the span of S to be the set of all possible finite linear combinations of vectors in S,

  $$[S] = \{a^1 v_1 + \cdots a^k v_k \ : \ \forall\, k \geq 0,\ a^1, \ldots, a^k \in \mathbb{R},\ v_1, \ldots, v_k \in S\}$$

- If $S = \{v_1, \ldots, v_m\}$ is linearly independent, then, for any $v \in [S]$, there exists a unique $\langle a^1, \ldots, a^k \rangle \in \widehat{R}^k$ such that

  $$v = a^1 v_1 + \cdots a^k v_k$$

- $[S]$ is a linear subspace of V

Basis and Dimension

- A vector space V is **finite dimensional** if there exists a finite set $S \subset V$ such that
$$[S] = V$$
- A **basis** of V is an ordered list of vectors, $E = (e_1, \ldots, e_m)$, such that
  - $S = \{e_1, \ldots, e_m\} \subset V$ is linearly independent
  - $[S] = V$
  - Equivalently, given any $v \in V$, there is a unique $\langle a^1, \ldots, a^m \rangle \in \widehat{R}^m$ such that
$$v = a^1 e_1 + \cdots + a^m e_m = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}$$
- Any finite dimensional vector space has at least one basis
- If $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_n)$ are both bases of V, then $m = n$ and the dimension of V is m

Basis of Abstract Vector Space

- A basis of a vector space $\mathbb{V}$ will be written as a row vector of vectors:
$$E = (e_1, \ldots, e_m) = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$$
- For each $v \in \mathbb{V}$, there is a unique column vector of scalars
$$a = \langle a^1, \ldots, a^m \rangle = \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}$$
  such that
$$\begin{aligned} v &= e_1 a^1 + \cdots + e_m a^m \\ &= \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = Ea \end{aligned}$$

Linear Isomorphism of a Vector Space with $\widehat{R}^m$

- Given a basis $E = (e_1, \ldots, e_m)$ of a vector space $\mathbb{V}$, there is an linear isomorphism

$$I_E : \widehat{R}^m \to \mathbb{V}$$
$$a = \langle a^1, \ldots, a^m \rangle \mapsto Ea = a^1 e_1 + \cdots + a^m e_m$$

Example: Basis of $\widehat{R}^m$

- Suppose $E = (e_1, \ldots, e_m)$ is a basis of $\widehat{R}^m$
- Each vector in E is of the form

$$e_k = \langle e_k^1, \ldots, e_k^m \rangle = \begin{bmatrix} e_k^1 \\ \vdots \\ e_k^m \end{bmatrix}$$

- Therefore, the basis can be written as a matrix:

$$E = \left[\begin{array}{c|c|c} e_1 & \ldots & e_m \end{array}\right] \left[\begin{array}{c|c|c} e_1^1 & \cdots & e_m^1 \\ \vdots & & \vdots \\ e_m^1 & \cdots & e_m^m \end{array}\right]$$

Vector in $\widehat{R}^m$ with Respect to Basis

- Given a basis E of $\widehat{R}^m$, as denoted above and a vector

$$v = \langle v^1, \dots, v^m \rangle \in \widehat{R}^m,$$

there exists a unique vector of coefficients

$$a = \langle a^1, \dots, a^m \rangle$$

such that

$$v = e_1 a^1 + \cdots + e_m a^m = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}$$

$$= \begin{bmatrix} e_1^1 & \cdots & e_m^1 \\ \vdots & & \vdots \\ e_m^1 & \cdots & e_m^m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = Ea$$

Standard Basis of $\widehat{R}^m$

- The standard basis of $\widehat{R}^m$ is

$$\begin{aligned} E &= (e_1, \dots e_m) \\ &= \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} e_1 &= \langle 1, 0, \dots, 0 \rangle \\ e_2 &= \langle 0, 1, \dots, 0 \rangle \\ &\vdots \\ e_m &= \langle 0, 0, \dots, 1 \rangle \end{aligned}$$

- Therefore, E can be written as a matrix

$$E = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = I \text{ (identity matrix)}$$

Change of Basis

- Let
$$E = (e_1, \ldots, e_m) \text{ and } F = (f_1, \ldots, f_m)$$
be bases of $\mathbb{V}$
- Since each f_k can be written as
$$f_k = e_1 A_k^1 + \cdots + e_m A_k^m,$$
there is a matrix A such that
$$F = EA$$
- Similary, there is a matrix B such that
$$E = FB$$
- Since
$$F = EA = FBA,$$
it follows that $BA = I$ and therefore $B = A^{-1}$

Change of coordinates

- Given a vector $v \in \mathbb{V}$, there exists a unique $a = I_E^{-1}(v) \in \widehat{R}^m$ such that
$$v = Ea$$
- Similarly, there exists a unique $b = I_F^{-1}(v) \in \widehat{R}^m$ such that
$$v = Fb$$
- If $F = EA$, then $E = FA^{-1}$ and therefore
$$Fb = v = Ea = FA^{-1}a$$
- It follows that
$$b = A^{-1}a$$

Example of Change of Basis

- The standard basis of $\widehat{R}^3$ is

$$E = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

- Consider another basis

$$\begin{aligned} f_1 &= \langle 1, 0, 0 \rangle = e_1 \\ f_2 &= \langle -2, 1, 0 \rangle = -2e_1 + e_2 \\ f_3 &= \langle 0, 1, 1 \rangle = e_2 + e_3 \end{aligned}$$

- This can also be written as

$$F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} = \left[\begin{array}{c|c|c} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

$$= E \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse basis

- Inverting

$$
\begin{aligned}
f_1 &= \langle 1, 0, 0 \rangle = e_1 \\
f_2 &= \langle -2, 1, 0 \rangle = -2e_1 + e_2 \\
f_3 &= \langle 0, 1, 1 \rangle = e_2 + e_3
\end{aligned}
$$

we get

$$
\begin{aligned}
e_1 &= f_1 \\
e_2 &= f_2 + 2e_1 = 2f_1 + f_2 \\
e_3 &= f_3 - e_2 = 2f_1 + f_2 + f_3
\end{aligned}
$$

or

$$
E = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = F \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
$$

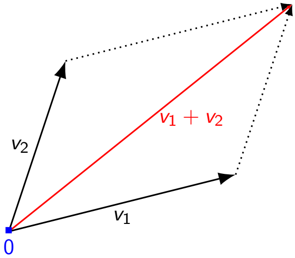
Change of coordinates

- Consider the vector $v = \langle 2, 3, -1 \rangle = 2e_1 + 3e_2 - e_3$
- To write it with respect to the basis F,

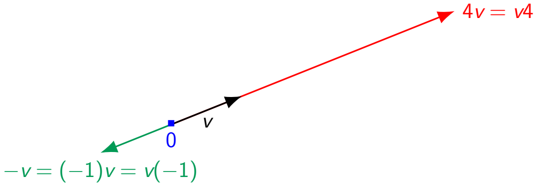
$$
\begin{aligned}
v = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} &= E \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \\
= F \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} &= F \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \\
= \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} &= 6f_1 + 2f_2 - f_3
\end{aligned}
$$

Geometric View of Vectors

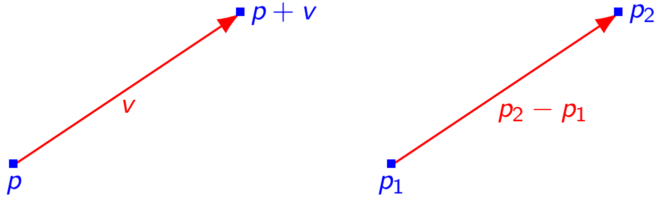
- Vector addition



- Scalar multiplication



Geometric View of Points and Vectors

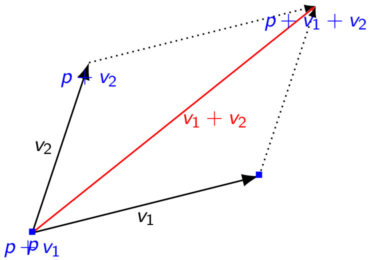


- A point p can be displaced by a vector v
- Given two points p_1 and p_2, there is a vector v such that p_2 is the point p_1 displaced by v

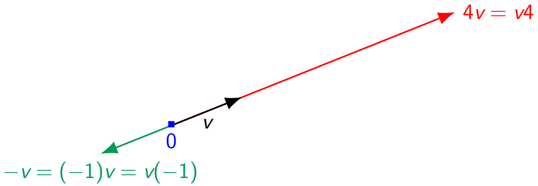
$$p_2 = p_1 + v \text{ and } v = p_2 - p_1$$

Geometric View of Points and Vectors

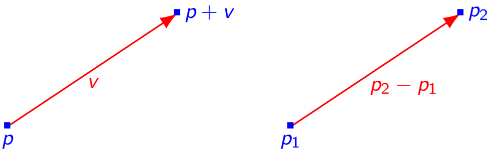
- Vector addition



- Scalar multiplication



Affine Space



- The set of all possible displacement vectors is a vector space $\mathbb{V}$, which we will call the *tangent space* of $\mathbb{A}$
- An affine space $\mathbb{A}$ with tangent space $\mathbb{V}$ is a set with the following operations:
  - Point-vector addition

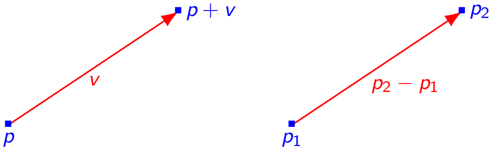
$$\begin{aligned} \mathbb{A} \times \mathbb{V} &\to \mathbb{A} \\ (p, v) &\mapsto p + v \end{aligned}$$

  - Point-point subtraction

$$\begin{aligned} \mathbb{A} \times \mathbb{A} &\to \mathbb{V} \\ (p_1, p_2) &\mapsto p_2 - p_1 \end{aligned}$$

which satisfy the following properties

Properties of Point-Vector Addition and Point-Point subtraction



- Point-vector addition
  - If $p \in \mathbb{A}$ and $v \in \mathbb{V}$, then $p + v \in \mathbb{A}$
  - $p + (v_1 + v_2) = (p + v_1) + v_2$
  - $p + 0 = p$
- Point-point subtraction
  - If $p_1, p_2 \in \mathbb{A}$, then $p_2 - p_1 \in \mathbb{V}$
  - $(p + v) - p = v$
  - $p_1 + (p_2 - p_1) = p_2$

Affine Combination of Points

- Recall that a linear combination of vectors $v_1, \ldots, v_k \in \mathbb{V}$ is of the form

$$a^1 v_1 + \cdots + v^k v_m, \text{ where } \langle a^1, \ldots, a^k \rangle \in \widehat{R}^k$$

- An affine combination of points $p_0, \ldots, p_k \in \mathbb{A}$ is a point of the form

$$p = p_0 + a^1(p_1 - p_0) + \cdots + a^k(p_k - p_0), \text{ where } \langle a^0, \ldots, a^k \rangle \in \widehat{R}^k$$

Affine Span

- Recall that the linear span of a subset $S \subset \mathbb{V}$ is

$$[S] = \{a^1 v_1 + \cdots + a^k v_k \ : \ v_1, \ldots, v_k \in S, \ k > 0\}$$

- The affine span of a subset $P \subset \mathbb{A}$ is the set of all possible affine combinations of finite subsets of P

$$[P] = \{p_0 + a^1(p_1 - p_0) + \cdots + a^k(p_k - p_0) \ : \ p_0, \ldots, p_k \in P, k \geq 0\}$$

- Given $P \subset \mathbb{A}$ and $p_0 \in P$, define

$$P - p_0 = \{p - p_0 \ : \ p \in P\} \subset \mathbb{V},$$

- Given $S \subset \mathbb{V}$ and $p_0 \in \mathbb{A}$, define

$$p_0 + S = \{p_0 + v \ : \ v \in \mathbb{V}\}$$

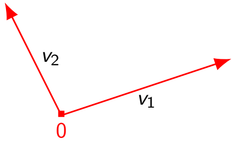
- If $p_0 \in P$,

$$[P] = p_0 + [P - p_0]$$

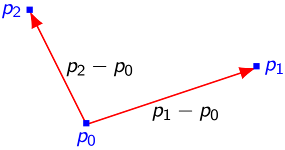
Affine Independence

- Recall that a set of vectors $\{v_1, \ldots, v_k\} \subset \mathbb{V}$ is linearly independent if the following holds:

$$a^1 v_1 + \cdots a^k v_k = 0 \implies a^1 = \cdots = a^k = 0$$

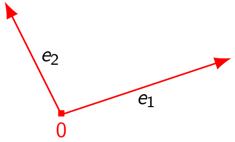


- A set of points $\{p_0, \ldots, p_k\} \subset \mathbb{A}$ is *affinely independent* if the set of vectors $\{p_1 - p_0, \ldots, p_k - p_0\}$ is linearly independent

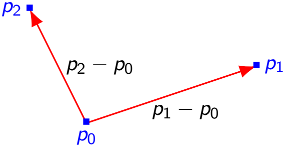


Affine Basis

- Recall that an ordered set of vectors $E = (e_1, \ldots, e_m)$ is a basis of $\mathbb{V}$ if the set $\{e_1, \ldots, e_m\}$ is linearly independent and the span of E is all of $\mathbb{V}$

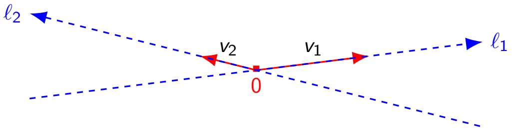


- An ordered set of points $P = (p_0, \ldots, p_m)$ is an *affine basis* of $\mathbb{A}$ if the ordered set of vectors $E = (p_1 - p_0, \ldots, p_m - p_0)$ is a basis of $\mathbb{V}$



- If the number of points in an affine basis is $m + 1$, then the dimension of the affine space is m

Geometry of Vector Space

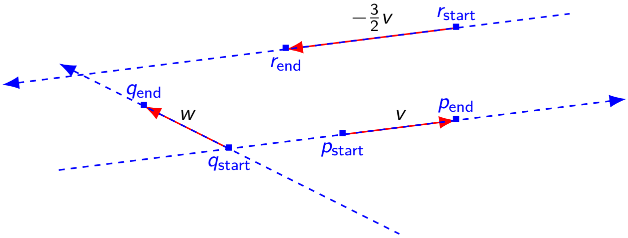


- A space of arrows or vectors
- The set of all scalar multiples of a nonzero vector is an oriented 1-dimensional linear subspace, i.e., an oriented line through the origin

$$\ell = \{tv \ : \ t \in \mathbb{R}\}$$

- A vector can be rescaled by a real number (called a scalar)
- Two vectors can be added using a parallelogram

Geometry of Affine Space



- A space of points
- From a point p_{start} to a different point p_{end} is a vector

$$v = p_{end} - p_{start}$$

- There is a unique oriented line passing through p_{start} and p_{end}

$$\ell = \{p_{start} + t(p_{end} - p_{start}) \ : \ t \in \mathbb{R}\}$$

- If $r_{end} - r_{start}$ is a scalar multiple of $p_{end} - p_{start}$, then the line through r_{start} and r_{end} is parallel to the line through p_{start} and p_{end}

Geometry of 3-Dimensional Abstract Vector and Affine Spaces

- Abstract vector space
  - Arrows, lines and planes through the origin
  - Can measure relative lengths of parallel vectors
  - No consistent way to compare the length of two vectors pointing in different directions
  - No way to measure the angle between two vectors
- Abstract affine space
  - Points, lines, planes
  - Can measure distance between two points relative to the distance between two other points only if all lie on a line
  - No consistent way to compare the distance between two pairs of points that lie on two non-parallel lines
  - No way to the measure the angle between two intersecting lines
- Measurement of lengths, distances, angles require something more