MATH-UA 377 Differential Geometry
Deductive Logic
Abstract Linear Algebra

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START RECORDING
LIVE TRANSCRIPT

Prerequisites: Deductive Logic

- Correct mathematical syntax, in both words and symbols
- Difference between mathematical sentence and mathematical phrase
- Rigorous meaning of
  - and
  - or
  - not
  - if...then...
  - ...if and only if...
- If A and B are mathematical sentences, the difference, if any, between the following:
  - If A, then B
  - If B, then A
  - If not A, then not B
  - If not B, then not A
  - A if and only if B

Logical Deduction

- Proof or calculation using modus ponens:
  - Suppose you know that the following sentences are true:
    - A
    - If A, then B
  - It follows that B is true
- Proof by contradiction
- Grading criteria include clarity of writing and logic
  - Incorrect use of logic will be penalized heavily
  - Skipping steps will be penalized

Logically Correct Calculations

- When doing a line-by-line calculation, you are expected to indicate clearly the logical flow
- Compare

$$\begin{aligned} x^2 &= 4 \\ \implies \quad x &= 2 \end{aligned}$$

to

$$\begin{aligned} x^2 &= 4 \\ \impliedby \quad x &= 2 \end{aligned}$$

and

$$\begin{aligned} x^2 &= 4 \\ \iff \quad x &= 2 \end{aligned}$$

Course Web Site

- https://www.math.nyu.edu/~yangd/MATH-UA0377Spring2022
- You are required to have read and consult regularly this web page and pages linked to it
- "I did not see ...on the web site" is never a valid excuse
- Follow instructions to sign up for
  - Gradescope
  - Overleaf

Course assignments

- All quizzes, homework assignments, and exams will be handled using Gradescope
- Quizzes
  - Questions will be true/false, multiple choice, or short answer
  - Answers are entered directly online
- Homework
  - Provided as Overleaf project
  - Solutions must be typed up using LaTeX
  - Solutions uploaded as PDF to Gradescope
- Midterm and final

Course Grade

- Quizzes: 5%
- Homework: 40%
- Midterm: 25%
- Final: 30%
- Plus tweaks

Overall Approach

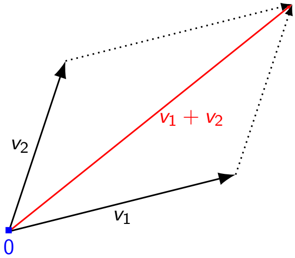
- Geometric properties and physical laws
  - Should not depend on the units of length and distance
  - Should not depend on the coordinates used for locations in space
- Ways to verify that a formula or property is geometric:
  - Check that it is invariant under change of coordinates
  - Use coordinates that are uniquely determined by the geometric assumptions
  - Write and verify it without using coordinates at all

Abstract Vector Space

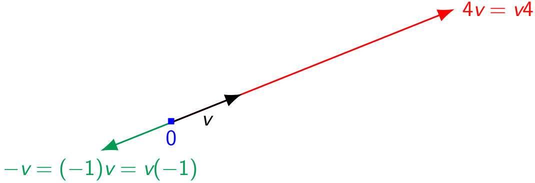
- A (real) vector space is a set V with two operations:
  - Vector addition
  - Scalar multiplication
- These operations must satisfy the following properties

Geometric View of Vectors

- Vector addition



- Scalar multiplication



Properties of Vector Addition

- Notation

$$V \times V \to V$$
$$(v_1, v_2) \mapsto v_1 + v_2,$$

- Associativity

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_2)$$

- Commutativity

$$v_1 + v_2 = v_2 + v_1$$

- Identity element: There exists an element $0 \in V$ such that, for any $v \in V$,

$$v + 0 = v$$

- Inverse element: For each $v \in V$, there exists an element, written as $-v$, such that

$$v + (-v) = 0$$

Scalar Multiplication

- Properties
  - Notation
$$\begin{aligned} \mathbb{R} \times V &\to V \\ (r, v) &\mapsto rv = vr \end{aligned}$$
  - Associativity
$$(r_1 r_2)v = r_1(r_2 v)$$
  - Distributivity
$$\begin{aligned} (r_1 + r_2)v &= r_1 v + r_2 v \\ r(v_1 + v_2) &= rv_1 + rv_2 \end{aligned}$$
  - Identity element
$$1v = v$$
- Consequences
$$\begin{aligned} 0v &= v \\ (-1)v &= v \end{aligned}$$

Examples

- The most important example is $\widehat{\mathbb{R}}^m$, whose elements are of the form

$$v = \langle r^1, \ldots, r^m \rangle = \begin{bmatrix} r^1 \\ \vdots \\ r^m \end{bmatrix},$$

  where $r^1, \ldots, r^m \in \mathbb{R}$
- The set of all solutions to

$$\begin{aligned} 3x - y + 2z &= 0 \\ x + 2y - z &= 0 \end{aligned}$$

- Polynomials of degree 10 or less
- Polynomials of any degree
- Continuous functions $f\colon \mathbb{R} \to \mathbb{R}$

Why Abstract Vector Spaces Are Useful

If we can prove something using only the properties of an abstract vector space, then it has to be true for any specific vector space

Basis of Abstract Vector Space

- A basis of a vector space $\mathbb{V}$ will be written as a row vector of vectors:
$$E = (e_1, \ldots, e_m) = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}$$
- For each $v \in \mathbb{V}$, there is a unique column vector of scalars
$$a = \langle a^1, \ldots, a^m \rangle = \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix}$$
such that
$$\begin{aligned} v &= e_1 a^1 + \cdots + e_m a^m \\ &= \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} \begin{bmatrix} a^1 \\ \vdots \\ a^m \end{bmatrix} = Ea \end{aligned}$$