f function

(1) $\boxed{df = \partial_x f\, dx + \partial_y f\, dy + \partial_z f\, dz}$

1-form

$\theta = \partial_? dx + \partial_? dy + \partial_3 dz$

$a\,dx + b\,dy + c\,dz$

$\boxed{d\theta = da \wedge dx + db \wedge dy + dc \wedge dz}$

$d(df) =$

$dx \wedge dy = -dy \wedge dz$

$dx \wedge dx = 0$

$\boxed{d(df) = 0 \iff \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$

$d^2 = 0 \iff$ "partials commute"

$\to df = \partial_1 f\, dx^1 + \partial_2 f\, dx^2 + \partial_3 f\, dx^3$

$$\theta = a_1 dx^1 + a_2 dx^2 + a_3 dx^3$$

$$d\theta = \boxed{da_1 \wedge} dx^1 + da_2 \wedge dx^2 + da_3 \wedge dx^3$$

a_1 function

$$da_1 = \partial_1 a_1 dx^1 + \partial_2 a_1 dx^2 + \partial_3 a_1 dx^3$$

$$= (\partial_1 a_1 dx^1 + \partial_2 a_1 dx^2 + \partial_3 a_1 dx^3) \wedge dx^1$$

$+ \cdot$ — — — —

$$= \cancel{(\partial_1 a_1 dx^1) \wedge dx^1} + (\partial_2 a_1 dx^2) \wedge dx^1 + (\partial_3 a_1 dx^3) \wedge dx^1$$

$+ \cdot$ — — — —

$$= \boxed{\partial_2 a_1 dx^2 \wedge dx^1} + \partial_3 a_1 dx^3 \wedge dx^1$$

$+ \quad +$ — — — $\boxed{\partial_1 a_2 dx^1 \wedge dx^2} +$

$+ \quad +$ —

$$= \partial_2 a_1 (-dx^1 \wedge dx^2) + \partial_1 a_2 \, dx^1 \wedge dx^2$$

$$\boxed{\begin{aligned} &= (\partial_1 a_2 - \partial_2 a_1)\, dx^1 \wedge dx^2 \\ &+ (\qquad\qquad)\, dx^3 \wedge dx^1 \\ &+ (\qquad\qquad)\, dx^2 \wedge dx^3 \end{aligned}}$$

$$\partial_1 \partial_3 \partial_2 a = \partial_1 \partial_2 \partial_3 a$$
$$= \partial_2 \partial_1 \partial_3 a$$

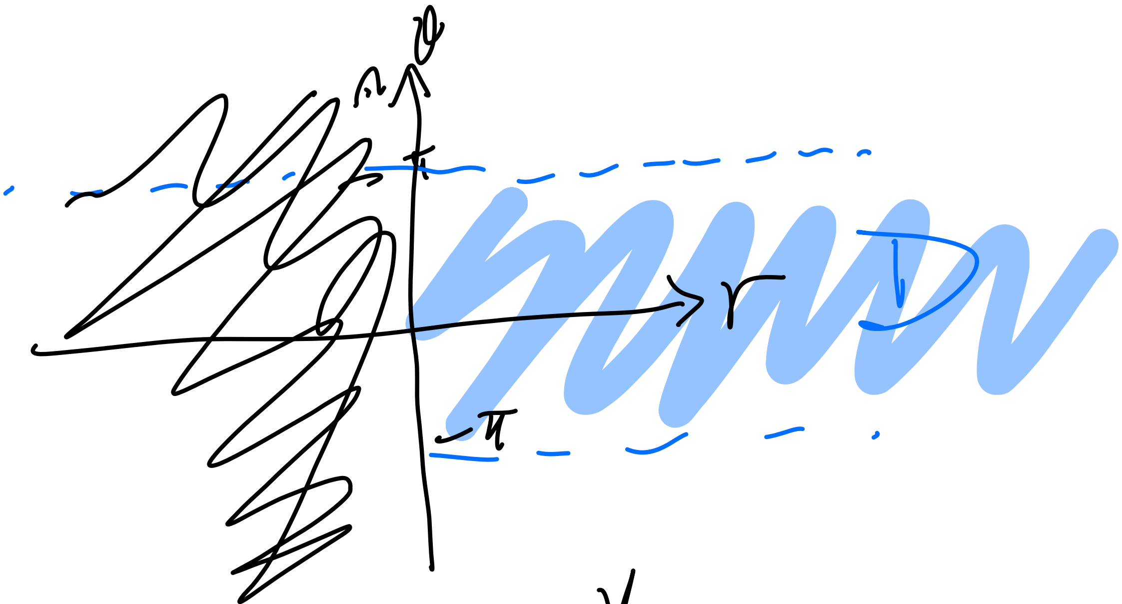
3.1

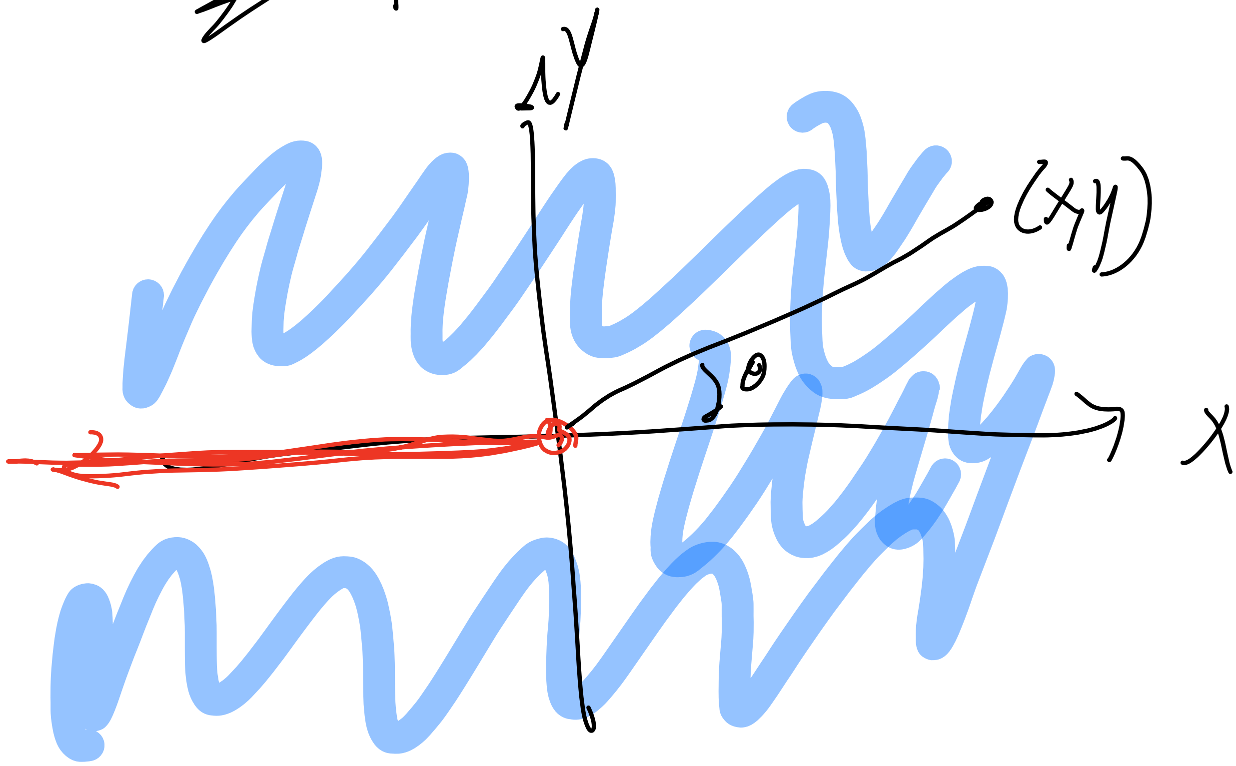
$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta)$$

$$\partial_r \Phi, \; \partial_\theta \Phi$$

$$\partial\Phi = \begin{bmatrix} \partial_r \Phi & \partial_\theta \Phi \end{bmatrix}$$ 2-by-2 matrix

nondegenerate means $\det \partial\Phi \neq 0$





$$-\pi < \theta < \pi$$

$$\Phi(r,\theta) = (\underbrace{r\cos\theta}_{= x},\ \underbrace{r\sin\theta}_{= y})$$

$$\boxed{\eta = \frac{-y\,dx + x\,dy}{x^2+y^2}}$$

$$dx = dr\cos\theta - r\sin\theta\, d\theta$$

$$dy = dr\sin\theta + r\cos\theta\, d\theta$$

$$\eta = \frac{(-r\sin\theta)(dr\cos\theta - r\sin\theta\, d\theta) + (r\cos\theta)(\quad)}{r^2}$$

$$= \frac{(\qquad)r + (\qquad)d\theta}{r^2}$$

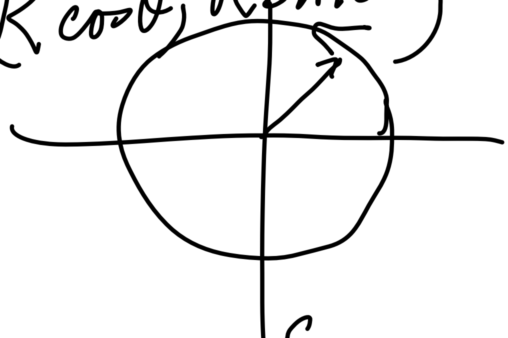
$\Phi^{*}\eta$ is a 1-form in (r,θ) space

$\parallel$

$$(\underline{\qquad})dr + (\qquad)d\theta$$

formula in r and θ formula in r and θ

$c(\theta) = (R\cos\theta, R\sin\theta)$



For a circle

$r = R$

$0 \le \theta \le 2\pi$

$$\int_c \eta = \int$$

$$\boxed{\eta = \frac{-y\,dx + x\,dy}{x^2+y^2}}$$

$$c(t) = (\underbrace{x(t)}, \underbrace{y(t)})$$

$$dx = x'dt$$
$$dy = dy'dt$$

$$c(\theta) = (\underset{\substack{\| \\ x}}{R\cos\theta}, \underset{\substack{\| \\ y}}{\boxed{R\sin\theta}})$$

$$dx = -R\sin\theta\, d\theta$$
$$dy = R\cos\theta\, d\theta$$

~~If~~ C is closed then $\int_C df = 0$

If C is closed and $\eta = df$, then $\int_C \eta = \int_C df = 0$

Surface parameterised by rectangle

$f = f(x,y,z)$ function

$d(\underbrace{df}_{\text{1-form}})$

$df = \partial_x f\, dx + \partial_y f\, dy + \partial_z dz$

$\theta = a\, dx + b\, dy + c\, dz$

$d\theta = \boxed{da} \wedge dx + \boxed{db} \wedge dx + \boxed{dc} \wedge dz$

a, b, c functions

$da = \partial_x a\, dx + \partial_y a\, dy + \partial_z a\, dz$

with

$d(df$, df 1-form

1) Put formula for df into $d(df)$

$d\left(\overbrace{\partial_x f}^{a}\, dx + \overbrace{\partial_y f}^{b}\, dy + \underbrace{\partial_z f}_{c}\, dz\right)$

$= d(\partial_x f) \wedge dx + d(\partial_y f) \wedge dy$

$d(\partial_x f) = \partial_x(\partial_x f)\, dx + \partial_y(\partial_x f)\, dy + \partial_z(\partial_x f)\, dz$

$$\partial_x(\partial_y f) = \partial_y(\partial_x f)$$

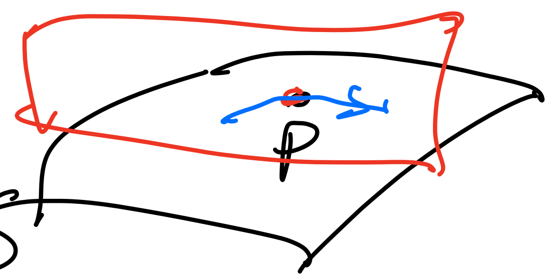
1) $S \subset \cancel{A^3}\ \mathbb{R}^3$

2) $R = (a,b) \times (c,d) \subset \bar{R} = [a,b] \times [c,d] \subset \mathbb{R}^3$

3) $\Phi : \bar{R} \longrightarrow \mathbb{R}^3$
coordinate map on R

Tangent space

$\partial_1 \Phi, \partial_2 \Phi$ basis of $T_p S$, each $p \in S$



Orientation of surface given by a basis (b_1, b_2) of $T_p S$ for every $p \in S$, depends continuously on $p \in S$

Coordinate map itself determines an orientation
$(b_1, b_2) = (\partial_1 \Phi(p), \partial_2 \Phi(p))$

If $(\partial_1\Phi, \partial_2\Phi)$ is not the one we want, just switch the order of variables or do calculation and multiply by -1 afteward

Integral of a 2-form on an oriented surface parameterized by rectangle

Assume $(\partial_1\Phi, \partial_2\Phi)$ is the right orientation

ω 2-form on S

Want $\int_S \omega$

Define

$$\int_S \omega = \boxed{\int_{R} \Phi^* \omega}$$

$\Phi^*\omega$ is a 2-form on R

R has standard orientation (∂_1, ∂_2)

$\cap$

$\mathbb{R}^2$

(consistent with $(\partial_1\Phi, \partial_2\Phi)$)

Concrete description

Coordinates on $\mathbb{R}^3$: (x,y,z)

Coordinates on $R \subset \mathbb{R}^2$: (u,v)

$$\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$$

$$\omega = a\, dy \wedge dz + b\, dz \wedge dx + c\, dx \wedge dy$$

a, b, c functions on $\cancel{R}$ $\mathbb{R}^3$

$\Phi^*\omega = \omega$ with x, y, z as functions of u, v

$$\Phi^* dx = \partial_u x\, du + \partial_v x\, dv$$

$\Phi^* dy$, $\Phi^* dz$ similar

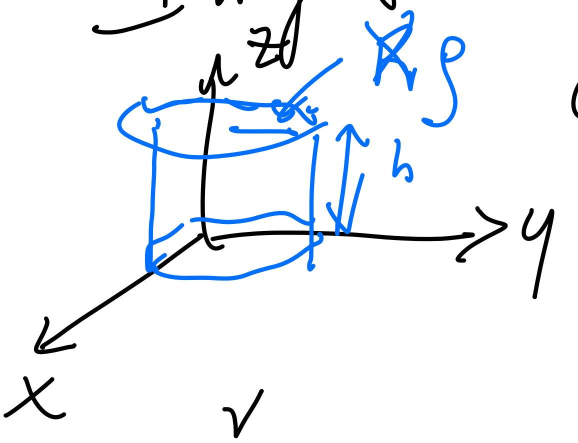
$$\Phi^*\omega = a(x(u,v), y(u,v), z(u,v))(\partial_u y\, du + \partial_v y\, dv) \wedge (\partial_u z\, du + \partial_v z\, dv)$$

$$\int_S \omega = \int_R \Phi^* \omega = \int_R (\qquad) \boxed{du\, dv}$$

$$= \int_{u=a}^{u=b} \int_{v=c}^{v=d} (\qquad)\, dv\, du$$

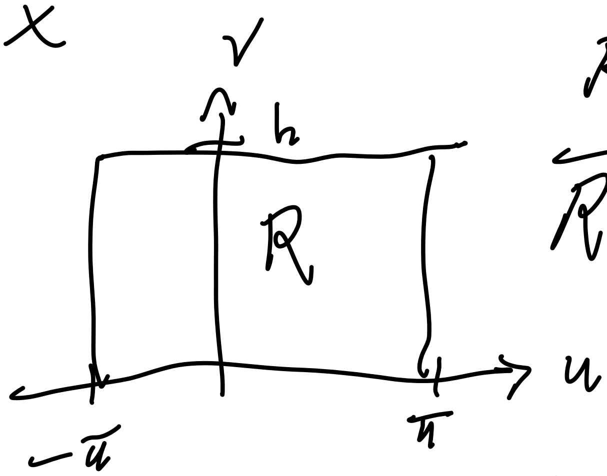
Must be in the right order

Integral over cylinder



$$C = \{ x^2 + y^2 = \cancel{R^2}\ \rho^2,\ 0 \le z \le h \}$$

ω



$$R = (-\pi, \pi) \times (0, h)$$

$$\overline{R} = [-\pi, \pi] \times [0, h]$$

Use cylindrical coordinates

$$\Phi(u,v) = (R\cos u, R\sin u, v)$$

$x = \rho\cancel{R}\cos u \qquad -\pi \le u \le \pi$

$y = \rho\cancel{R}\sin u \qquad 0 \le v \le h$

$z = v$

Φ is coordinate on R

but is not bijective on $\overline{R}$

OK if Φ is not bijective on ∂R

Want compute this example called (ii) earlier

$$\int_C \boxed{y\, dx \wedge dz}$$

$$\parallel$$

$$\int_R \Phi^*(y\, dx \wedge dz)$$

$x = \rho\cos u \Rightarrow dx = -\rho \sin u\, du$

$y = \rho \sin u \Rightarrow dy = \ldots$

$z = v \Rightarrow dz = dv$

$\Rightarrow \Phi^*(y\, dx \wedge dz) = (\rho\sin u)(-\rho \sin u\, du) \wedge dv$

$$\Phi^*(y\,dx \wedge dz) = -\rho^2(\sin u)^2\, du \wedge dv$$

correct order

$$\Rightarrow \int_C y\,dx \wedge dz = \int_R \Phi^*(y\,dx \wedge dz)$$

$$= \int_R -\rho^2(\sin u)^2\, du \wedge dv$$

$$= \int_R -\rho^2(\sin u)^2\, du\, dv$$

$$= \boxed{\int_{u=-\pi}^{u=\pi}\int_{v=0}^{v=h} -\rho^2(\sin u)^2\, du\, dv}$$

$$= -\rho^2\left(\int_{u=-\pi}^{u=\pi}(\sin u)^2\, du\right)\left(\int_{v=0}^{v=h} dv\right)$$

$$(\sin u)^2 = \frac{1-\cos 2\theta}{2}$$

$$= -\rho^2\left(\int_{u=-\pi}^{u=\pi}\frac{1-\cos 2\theta}{2}\, d\theta\right)h$$

Stokes' Theorem on Parameterized Surface

Assume ω is 1-form on $\mathbb{R}^3$

$$\omega = p\,dx + q\,dy + r\,dz$$

S surface parameterized by rectangle

$$\int_S d\omega = \int_R \Phi^*(d\omega) = \int_R d(\Phi^*\omega)$$

Recall Stokes' theorem on rectangle R

$$\int_R \underbrace{d\theta}_{\text{1-form on } R} = \underbrace{\int_{\partial R} \theta}_{\text{line integral}}$$

2-form

1-form

$$\int_R d(\Phi^* \omega) = \int_R d\theta \text{ where } \theta = \Phi^* \omega$$

By Stokes'

$$\int_R d(\Phi^* \omega) = \underbrace{\int_{\partial R} \Phi^* \omega}_{\text{line integral}} = \underbrace{\int_{\partial S} \omega}_{\text{line integral}}$$

$$\boxed{\int_S d\omega = \int_{\partial S} \omega}$$

Stokes' theorem on a rectangle.

Key facts: $d(\Phi^* \omega) = \Phi^*(d\omega)$

$$\omega = a\,dx + by + c\,dz$$

$$\Phi^* \omega = a(x_u du + x_v dv) + \cdots$$

$$a(x(u,v), y(u,v), z(u,v))$$

$$= (a\partial_u x + b\partial_u y + c\partial_u z)\,du$$

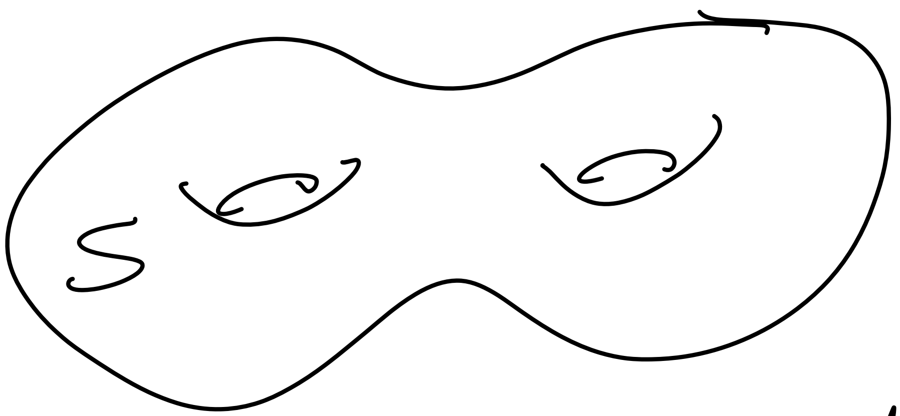
$$+ (\qquad) \, dv$$

$$d(\Phi^*\omega) = \partial_v(a\partial_u x + b\partial_u y + c\partial_u z)\,dv\wedge du$$
$$+\ \partial_u(\qquad)\,du\wedge dv$$

$$d\omega = da\wedge dx + db\wedge dy + dc\wedge dz$$
$$= \dots = (\qquad)\,dy\wedge dz$$
$$+\ (\qquad)\,dz\wedge dx$$
$$+\ (\qquad)\,dx\wedge dy$$

$$\Phi^*(d\omega) = \dots$$

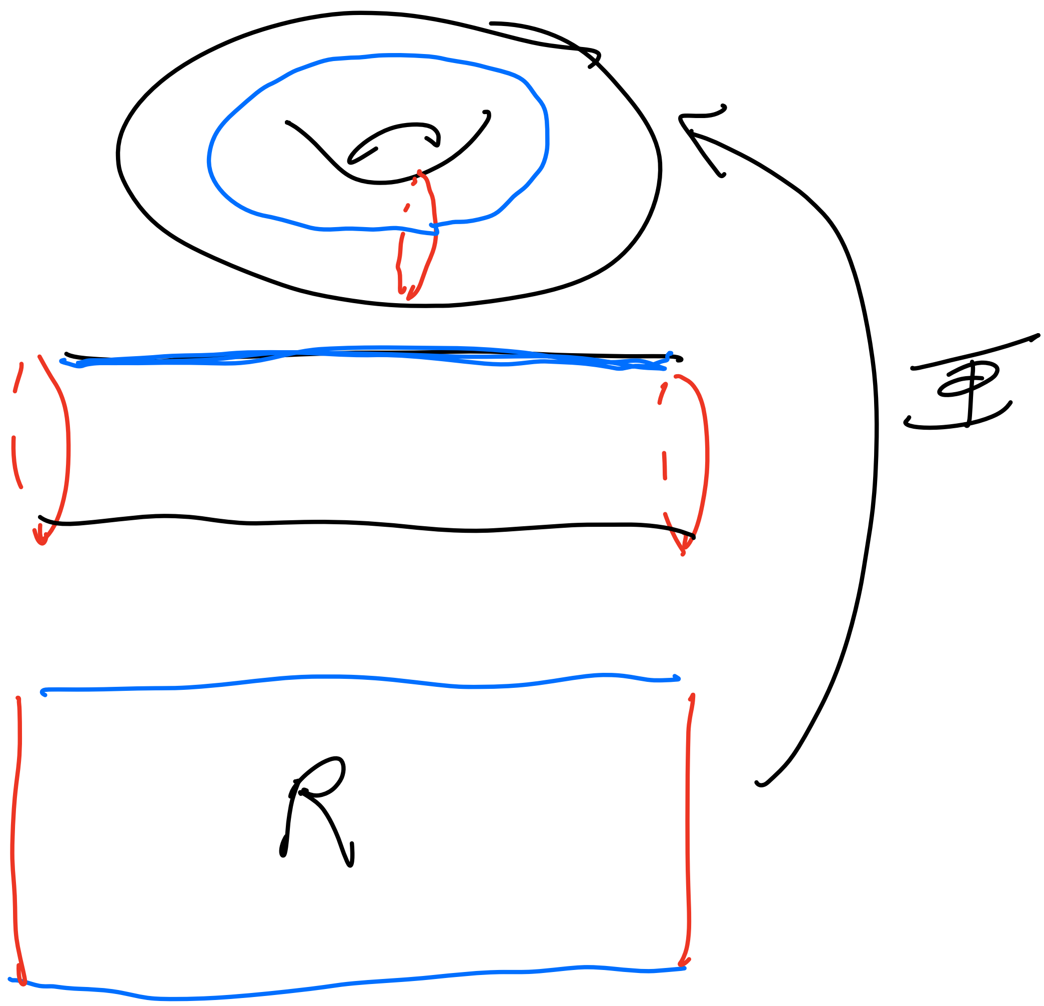
Are equal after simplified



S = oriented surface

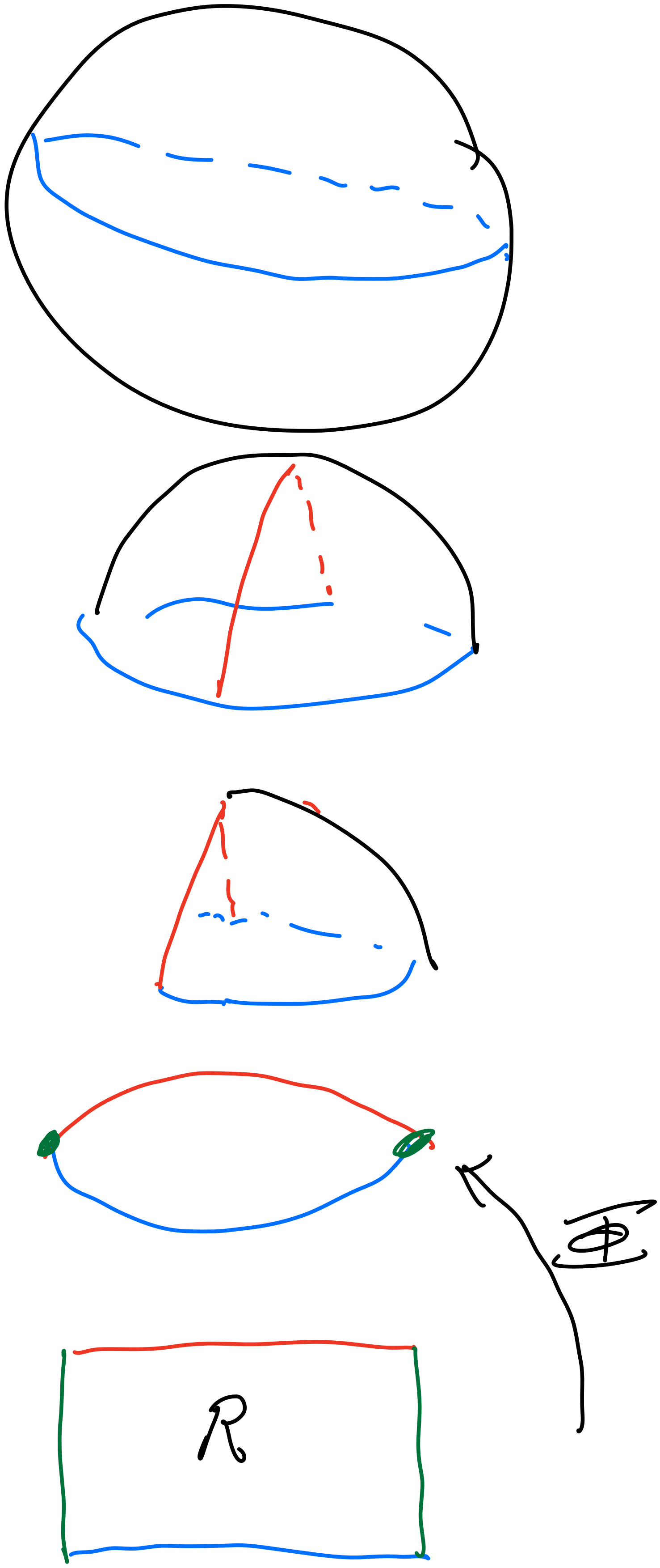
$$\int_S \omega$$

Idea: Chop S into rectangles



Φ bijective on R
but not on $\overline{R}$

Sphere



Suppose $S = S_1 \cup S_2 \cup \cdots \cup S_N$

$\Phi_k : R_k \to S_k$

Given 2-form ω on S,

define

$$\int_S \omega = \sum_{k=1}^{N} \int_{S_k} \omega = \sum_{k=1}^{N} \int_{R_k} \Phi_k^* \omega$$

Key requirement:

1) S oriented

2) Orientation of each $(\partial_1 \Phi_k, \partial_2 \Phi_k)$ matches orientation of S

Spherical coordinates

$$\Phi(\varphi, \theta) = (\cancel{\rho} \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)$$

$0 \leq \varphi \leq \pi, \quad 0 \leq \theta \leq 2\pi$

$\Phi : [0, \pi] \times [0, 2\pi] \to S$

Φ bijective on open rectangle

$$\int_S z^2\, dx \wedge dy$$

$z = \rho\cos\varphi$

$dx = \rho\cos\varphi\cos\theta\, d\varphi - \rho\sin\varphi\sin\theta\, d\theta$

$dy = \rho\cos\varphi\sin\theta\, d\varphi + \rho\sin\varphi\cos\theta\, d\theta$

$\Phi^*(dx \wedge dy) = (\qquad\qquad)\, d\varphi \wedge d\theta$

Use orientation given by $(\partial_\varphi\Phi, \partial_\theta\Phi)$

$$\int_{\varphi=0}^{\varphi=\pi}\int_{\theta=0}^{\theta=2\pi} (\qquad\qquad)\, d\varphi\, d$$