MATH-UA 123 Calculus 3: Review

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START RECORDING

Key Topics to Review

- Equation of plane
  - $ax + by + cz = d$
  - Using normal vector and point
  - Using two vectors parallel to plane and point
  - Using three points
  - From equation to normal
- Normal vector to surface
  - Normal to contour surface
  - Normal to graph
- Tangent plane to surface
- Differential of a function
  - Linear approximation
  - Chain rule
- Identification of critical point type
  - Using contours
  - Using second derivative type
- Constrained optimization
  - Using contours
  - Using Lagrange multipliers

Surface Area of a Surface

- The surface area of a surface S is equal to

$$\text{Area} = \int_S dA$$

- To compute this, recall that

$$d\vec{S} = \vec{n}\, dA$$

and therefore

$$dA = \vec{n} \cdot (\vec{n}\, dA) = \vec{n}\, d\vec{S}$$

- In other words, the surface area is equal to the flux of $\vec{n}$ through S

$$\text{Area} = \int_S \vec{n} \cdot d\vec{S}$$

- Using a parameterization $\vec{r}(s,t)$, $(s,t) \in D$, of S,

$$\begin{aligned}
\text{Area} &= \int_S \vec{n} \cdot d\vec{S} \\
&= \int_S \vec{n} \cdot (\vec{r}_s \times \vec{r}_t)\, ds\, dt \\
&= \int_S \left(\frac{\vec{r}_s \times \vec{r}_t}{|\vec{r}_s \times \vec{r}_t|} \right) \cdot (\vec{r}_s \times \vec{r}_t)\, ds\, dt \\
&= \int_S |\vec{r}_s \times \vec{r}_t|\, ds\, dt
\end{aligned}$$

Example: Surface Area of Sphere with Radius R

$$
\begin{aligned}
\vec{r}(\phi,\theta) &= R\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle \\
\vec{r}_\phi \times \vec{r}_\theta &= R^2 \sin\phi \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle \\
&= R\sin\phi\,\vec{r} \\
|\vec{r}_\phi \times \vec{r}_\theta| &= R^2\sin\phi \\
\text{Area} &= \int_{\phi=0}^{\phi=\pi}\int_{\theta=0}^{\theta=2\pi} R^2\sin\phi\, d\theta\, d\phi \\
&= R^2\int_{\phi=0}^{\phi=\pi}\sin\phi\, d\phi \int_{\theta=0}^{\theta=2\pi} d\theta \\
&= 2\pi R^2\left(-\cos\phi|_{\phi=0}^{\phi=\pi}\right) \\
&= 2\pi R^2(-(-1)-(-1)) \\
&= 4\pi R^2
\end{aligned}
$$

Example: Surface Area of Circular Paraboloid

- Consider the surface

$$S = \{z = x^2 + y^2 \ : \ x^2 + y^2 \le R^2\}$$

- Parameterize S: Let $D = \{x^2 + y^2 \le R^2\}$

$$\begin{aligned}
\vec{r}(x, y) &= \langle x, y, x^2 + y^2 \rangle \\
|\vec{r}_x \times \vec{r}_y| &= |\langle 1, 0, 2x\rangle \times \langle 0, 1, 2y\rangle| = |\langle -2x, -2y, 1\rangle| = \sqrt{1 + 4x^2 + 4y^2} \\
\text{Area} &= \int_D |\vec{r}_x \times \vec{r}_y| \, dx\, dy \\
&= \int_D \sqrt{1 + 4x^2 + 4y^2} \, dx\, dy \\
&= \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} \sqrt{1 + 4r^2} (r\, dr\, d\theta) \\
&= \int_{r=0}^{r=R} \sqrt{1 + 4r^2}\, r\, dr \int_{\theta=0}^{\theta=2\pi} d\theta \\
&= 2\pi \int_{u=1}^{u=1+4R^2} u^{1/2} \frac{1}{8} \, du, \text{ where } u = 1 + 4r^2 \text{ and } du = 8r\, dr \\
&= \frac{1}{4}\pi \left(\frac{2}{3} u^{3/2} \Big|_{u=1}^{u=1+4R^2} \right) = \frac{1}{6}\pi((1 + 4R^2)^{3/2} - 1)
\end{aligned}$$

Fundamental Theorems of Calculus

- (Fundamental Theorem of Calculus)

$$\int_{t=a}^{t=b} f'(t)\, dt = f(b) - f(a)$$

- (Fundamental Theorem of Line Integrals) Given an oriented curve C from $\vec{r}_{\text{start}}$ to $\vec{r}_{\text{end}}$,

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}_{\text{end}}) - f(\vec{r}_{\text{start}})$$

- (Green's Theorem) Given a domain D in 2-space with positively oriented boundary ∂D:

$$\int_D \vec{\nabla} \times \vec{F}\, dA = \int_{\partial D} \vec{F} \cdot d\vec{r}$$

- (Divergence Theorem) Given a domain R with positively oriented boundary ∂R in 3-space,

$$\int_R \vec{\nabla} \cdot \vec{F}\, dV = \int_{\partial R} \vec{F} \cdot d\vec{S}$$

Computing Integrals

Space	Integral	Integrand	Domain	Method
$\mathbb{R}^2$	Double	Function	2D Region	Directly
$\mathbb{R}^2$ or $\mathbb{R}^3$	Line	Gradient field	Oriented curve	Fundamental Theorem of Line Integrals
$\mathbb{R}^2$ or $\mathbb{R}^3$	Line	Vector field	Oriented curve	Directly
$\mathbb{R}^2$	Line	Vector field	Closed oriented curve in $\mathbb{R}^2$	Directly or Green's Theorem
$\mathbb{R}^3$	Line	Vector field	Closed oriented curve	Directly or Stokes' Theorem
$\mathbb{R}^3$	Flux	Vector field	Oriented surface	Directly
$\mathbb{R}^3$	Flux	Vector field	Closed surface	Directly or Divergence Theorem
$\mathbb{R}^3$	Flux	Curl of a vector field	Closed surface	Directly or Divergence Theorem
$\mathbb{R}^3$	Triple	Divergence of a vector field	3D region	Directly or Divergence Theorem

Examples

Let $\vec{F} = \vec{i}x^3 + \vec{j}y^3 + \vec{k}z^3$
Compute the following:

- $\int_S \vec{F} \cdot d\vec{S}$, where S is the disk of radius 3 in the plane $y = 5$, oriented toward the origin
- $\int_W \vec{F}\, dV$, where W is the solid sphere of radius 2 centered at the origin
- $\int_S \vec{F} \cdot d\vec{S}$, where S is the sphere of radius 2 centered at the origin
- $\int_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$, where S is the disk of radius 3 in the plane $y = 5$, oriented toward the origin
- $\int_C \vec{\nabla}\vec{F} \cdot d\vec{r}$, where C is the line from the origin to $(2, 3, 4)$
- $\int_W \vec{\nabla} \cdot \vec{F}\, dV$, where W is the ball of radius 2 centered at the origin
- $\int_C \vec{F} \cdot d\vec{r}$, where C is the line segment from the origin to $(2, 3, 4)$

More Examples

- Let $\vec{F}$ be a vector field on 3-space
- Let S_1 be the upper half of the sphere of radius 1 centered at the origin, oriented upward
- Let S_2 be the disk of radius 1 in the xy-plane centered at the origin and oriented upward
- Let C be the unit circle in the xy-plane, oriented clockwise when viewed from above
- Let W be the half ball that lies between S_2 and S_1
- Which of the following integrals are equal?

$$\int_C \vec{F} \cdot d\vec{r} \qquad \int_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

$$\int_C (\vec{\nabla} \times \vec{F}) \cdot d\vec{r} \qquad \int_{S_2} \vec{F} \cdot d\vec{S}$$

$$\int_C \vec{F} \cdot d\vec{S} \qquad \int_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

$$\int_{S_1} \vec{F} \cdot d\vec{r} \qquad \int_W \vec{\nabla} \cdot \vec{F}\, dV$$