# MATH-UA 123 Calculus 3: Review

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**START RECORDING**

# Final Exam Coverage

- Chapter 10
  - 10.1: Coordinates
  - 10.2: Vectors
  - 10.3: Dot product, orthogonal projection
  - 10.4: Cross product
  - 10.5: Equations of lines and planes
  - 10.6: Cylinders and quadric surfaces
  - 10.7: Parameterized curves
  - 10.8: Arclength
  - 10.9: Position, velocity, speed, acceleration
- Chapter 11
  - 11.1: Functions of several variables
  - 11.3: Partial derivatives
  - 11.4: Tangent planes, linear approximations, differentials
  - 11.5: Chain rule
  - 11.6: Directional derivatives, gradient
  - 11.7: Maximum and minimum values
  - 11.8: Lagrange multipliers

# Final Exam Coverage

- Chapter 12
  - 12.1-12.2: Double integrals
  - 12.3: Polar coordinates
  - 12.5: Triple integrals
  - 12.6: Cylindrical coordinates
  - 12.7: Spherical coordinates
- Chapter 13
  - 13.1: Vector fields
  - 13.2: Line integrals
  - 13.3: Fundamental theorem of line integrals
  - 13.4: Green's Theorem
  - 13.5: Curl and divergence
  - 13.6: Parametric surfaces
  - 13.7: Flux integrals
  - 13.9: Divergence Theorem

# Chapter 10

- Cross product and curl: Compute quickly with minimal effort
- Equations of lines and planes
  - Using equations or parameterization
  - From normal vector to equation of plane and vice versa
  - From parameterization of plane to normal vector and vice versa
- Cylinders and quadric surfaces
  - Equation to parameterization and vice versa
  - Shape to equation and vice versa
- Parameterized curves
  - Description to parameterization and vice versa
  - Velocity, speed, acceleration
  - Length of curve = integral of speed

# Chapter 11

- Linear approximation
  - Equation of tangent plane
  - Use differentials to do linear approximation
- Chain rule
  - Using differentials or compute directly
- Gradient and directional derivatives
  - Gradient = direction of fastest increase
  - Directional derivative = dot product of direction and gradient
  - Gradient = conservative = path-independent vector field
  - Curl of gradient = 0
  - Curl test on simply connected domain
- Shape of graph near a critical point
  - Using contours
  - Using second derivative test
- Constrained optimization using Lagrange multipliers

# Chapter 12

- Double and triple integrals
  - Finding endpoints of integration
  - Switching order of integration
  - Using polar, cylindrical, or spherical coordinates
- Line integral
  - Using parameterization of curve
  - Using Fundamental Theorem of Line Integrals, when vector field is conservative
  - Using Green's Theorem
- Flux integral
  - Easy way when $\vec{F} \cdot n$ is constant on surface
  - As a double integral when the surface is flat and parallel to a coordinate plane
  - Using parameterization of surface
  - Using Divergence Theorem

# Advice

- The most important skill is not knowing how to solve a problem but knowing how to figure out how to solve a problem
- Review everything:
  - Examples from lectures, textbooks, anywhere else
  - Quizzes
  - Written homework problems
  - WebAssign problems
- Try to redo problems from scratch
  - Pretend you've never seen a problem before
  - Don't try to remember what the solution was
  - Try to see what possible paths there are to a solution
  - Work out the details of each possible path
- Peek at solutions only when you're totally lost and peek only just enough so that you can see what you need to do
- If the lecture notes and textbooks are confusing, consult other sources such as Khan Academy to see if their explanations are easier to understand

# Understanding Concepts Versus Memorizing Procedures

- The real goal: Do well on the final
- Understanding concepts is only a means to the end
- Memorization of what things are is important
- Memorization of procedures like computing a line or flux integral is important
- Conceptual understanding is important because it guides you in figuring out what things or procedures you need to use for solving a problem

# Fundamental Theorems of Calculus

- (Fundamental Theorem of Calculus)

$$\int_{t=a}^{t=b} f'(t)\,dt = f(b) - f(a)$$

- (Fundamental Theorem of Line Integrals) Given an oriented curve $C$ from $\vec{r}_{\text{start}}$ to $\vec{r}_{\text{end}}$,

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}_{\text{end}}) - f(\vec{r}_{\text{start}})$$

- (Green's Theorem) Given a domain $D$ in 2-space with positively oriented boundary $\partial D$:

$$\int_D \vec{\nabla} \times \vec{F}\,dA = \int_{\partial D} \vec{F} \cdot d\vec{r}$$

- (Divergence Theorem) Given a domain $R$ with positively oriented boundary $\partial R$ in 3-space,

$$\int_R \vec{\nabla} \cdot \vec{F}\,dV = \int_{\partial R} \vec{F} \cdot d\vec{S}$$

# Choices

| Integral | Integrand | Domain | Method |
|---|---|---|---|
| Line | Vector field | Oriented curve | Directly |
| Line | Gradient field | Oriented curve | Fundamental Theorem of Line Integrals |
| Double | Function | 2D Region | Directly or Green's Theorem |
| Flux | Vector field | Oriented surface in 3-space | Directly or Stokes' Theorem |
| Flux | Vector field | Closed surface in 3-space | Directly or Divergence Theorem |
| Triple | Divergence | 3D region | Directly or Divergence Theorem |

# Computing Integrals

| Integral | Integrand | Domain | Method |
|---|---|---|---|
| $\int_C \vec{F} \cdot d\vec{r}$ | $\vec{F} =$ vector field | $C =$ oriented curve with endpoints | Directly |
| $\int_C \vec{F} \cdot d\vec{r}$ | $\vec{F} =$ conservative field | $C =$ oriented curve | Fundamental Theorem of Line Integrals |
| $\int_C \vec{F} \cdot d\vec{r}$ | $\vec{F} =$ vector field | $C =$ closed oriented curve | Directly or Green's Theorem |
| $\int_D f\, dA$ | $f =$ scalar function | $D =$ 2D Region | Directly (or Green's Theorem) |
| $\int_S \vec{F} \cdot d\vec{S}$ | $\vec{F} =$ vector field | $S =$ oriented surface with boundary in 3-space | Directly (or Stokes' Theorem) |
| $\int_S \vec{F} \cdot d\vec{S}$ | $\vec{F} =$ vector field | $S =$ oriented closed surface in 3-space | Directly or Divergence Theorem |
| $\int_R f\, dV$ | $f =$ scalar function | $R =$ 3D region | Directly (or Divergence Theorem) |

## Examples

Let $\vec{F} = \vec{i}x^3 + \vec{j}y^3 + \vec{k}z^3$
Compute the following:

- $\int_S \vec{F} \cdot d\vec{S}$, where $S$ is the disk of radius 3 in the plane $y = 5$, oriented toward the origin
- $\int_W \vec{F}\, dV$, where $W$ is the solid sphere of radius 2 centered at the origin
- $\int_S \vec{F} \cdot d\vec{S}$, where $S$ is the sphere of radius 2 centered at the origin
- $\int_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$, where $S$ is the disk of radius 3 in the plane $y = 5$, oriented toward the origin
- $\int_C \vec{\nabla}\vec{F} \cdot d\vec{r}$, where $C$ is the line from the origin to $(2, 3, 4)$
- $\int_W \vec{\nabla} \cdot \vec{F}\, dV$, where $W$ is the ball of radius 2 centered at the origin
- $\int_C \vec{F} \cdot d\vec{r}$, where $C$ is the line segment from the origin to $(2, 3, 4)$

# More Examples

- Let $\vec{F}$ be a vector field on 3-space
- Let $S_1$ be the upper half of the sphere of radius 1 centered at the origin, oriented upward
- Let $S_2$ be the disk of radius 1 in the $xy$-plane centered at the origin and oriented upward
- Let $C$ be the unit circle in the $xy$-plane, oriented clockwise when viewed from above
- Let $W$ be the half ball that lies between $S_2$ and $S_1$
- Which of the following integrals are equal?

$$\int_C \vec{F} \cdot d\vec{r} \qquad \int_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

$$\int_C (\vec{\nabla} \times \vec{F}) \cdot d\vec{r} \qquad \int_{S_2} \vec{F} \cdot d\vec{S}$$

$$\int_C \vec{F} \cdot d\vec{S} \qquad \int_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

$$\int_{S_1} \vec{F} \cdot d\vec{r} \qquad \int_W \vec{\nabla} \cdot \vec{F} \, dV$$