MATH-UA 123 Calculus 3: Divergence Theorem

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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Fundamental Theorems of Calculus

- (Fundamental Theorem of Calculus)

$$\int_{t=a}^{t=b} f'(t)\, dt = f(b) - f(a)$$

- (Fundamental Theorem of Line Integrals) Given an oriented curve C from $\vec{r}_{\text{start}}$ to $\vec{r}_{\text{end}}$,

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}_{\text{end}}) - f(\vec{r}_{\text{start}})$$

- (Green's Theorem) Given a domain D in 2-space with positively oriented boundary ∂D:

$$\int_D \vec{\nabla} \times \vec{F}\, dA = \int_{\partial D} \vec{F} \cdot d\vec{r}$$

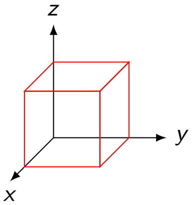
- (Stokes' Theorem) Given an oriented surface S in 3-space with positively oriented boundary ∂S:

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

- (Divergence Theorem) Given a domain R with positively oriented boundary ∂R in 3-space,

$$\int_R \vec{\nabla} \cdot \vec{F}\, dV = \int_{\partial R} \vec{F} \cdot d\vec{S}$$

Parameterization of Cube Boundary



- Let $C = [0,1] \times [0,1] \times [0,1]$
- The boundary of C is a union of sides

$$\partial C = S_{-\vec{i}} \cup S_{\vec{i}} \cup S_{-\vec{j}} \cup S_{\vec{j}} \cup S_{-\vec{k}} \cup S_{\vec{k}},$$

  where $S_{\vec{n}}$ is the side with outer unit normal $\vec{n}$
- The sides have the following parameterizations:

$$\begin{aligned}
S_{-\vec{i}}: \vec{r}(z,y) &= \langle 0, y, z\rangle,\ 0 \le y, z \le 1 \text{ (back)}\\
S_{\vec{i}}: \vec{r}(y,z) &= \langle 1, y, z\rangle,\ 0 \le y, z \le 1 \text{ (front)}\\
S_{-\vec{j}}: \vec{r}(x,z) &= \langle x, 0, z\rangle,\ 0 \le z, x \le 1 \text{ (left)}\\
S_{\vec{j}}: \vec{r}(z,x) &= \langle x, 1, z\rangle,\ 0 \le z, x \le 1 \text{ (right)}\\
S_{-\vec{k}}: \vec{r}(y,x) &= \langle x, y, 0\rangle,\ 0 \le y, x \le 1 \text{ (bottom)}\\
S_{\vec{k}}: \vec{r}(x,y) &= \langle s, t, 1\rangle,\ 0 \le x, y \le 1 \text{ (top)}
\end{aligned}$$

Flux Integrals Through Front and Back of Cube

- The outward flux of $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3$ through $S_{-\vec{i}} \cup S_{\vec{i}}$ is

$$\begin{aligned}
\int_{S_{-\vec{i}} \cup S_{\vec{i}}} \vec{F} \cdot d\vec{S} &= \int_{S_{-\vec{i}} \cup S_{\vec{i}}} \vec{F} \cdot \vec{n}\, dA \\
&= \int_{z=0}^{z=1} \int_{y=0}^{y=1} \vec{F}(0, y, z) \cdot (-\vec{i})\, dy\, dz + \int_{z=0}^{z=1} \int_{y=0}^{y=1} (\vec{F}(1, y, z) \cdot \vec{i})\, dy \\
&= \int_{z=0}^{z=1} \int_{y=0}^{y=1} F_1(1, y, z) - F_1(0, y, z)\, dy\, dz \\
&= \int_{z=0}^{z=1} \int_{y=0}^{y=1} \int_{x=0}^{x=1} (F_1)_x(x, y, z)\, dx\, dy\, dz \\
&= \int_C (F_1)_x\, dV
\end{aligned}$$

- Repeating this for the other sides, the outward flux of $\vec{F}$ through ∂C is

$$\begin{aligned}
\int_{\partial C} \vec{F} \cdot d\vec{S} &= \int_{S_{-\vec{i}} \cup S_{\vec{i}}} \vec{F} \cdot d\vec{S} + \int_{S_{-\vec{j}} \cup S_{\vec{j}}} \vec{F} \cdot d\vec{S} + \int_{S_{-\vec{k}} \cup S_{\vec{k}}} \vec{F} \cdot d\vec{S} \\
&= \int_C (F_1)_x + (F_2)_y + (F_3)_z\, dV = \int_C (\vec{\nabla} \cdot \vec{F})\, dV
\end{aligned}$$

Divergence Theorem

- The divergence of a vector field $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3$ is the scalar function given by

$$\vec{\nabla} \cdot \vec{F} = (F_1)_x + (F_2)_y + (F_3)_z$$

- We have shown that, if C is a cube, ∂C its boundary with the outward orientation, and $\vec{F}$ is a vector field on C, then

$$\int_C \vec{\nabla} \cdot \vec{F} \, dV = \int_{\partial C} \vec{F} \cdot d\vec{S}$$

- Any 3-dimensional region R can be chopped up into pieces, where each can be parameterized by a cube

Theorem (Divergence Theorem)

Suppose R is a 3-dimensional region and ∂R is the boundary with the outward orientation. If $\vec{F}$ is a vector field on R, then

$$\int_R (\vec{\nabla} \cdot \vec{F}) \, dV = \int_{\partial R} \vec{F} \cdot d\vec{S}$$

Flux Out Of Cylinder using the Divergence Theorem

- Suppose R is the solid circular cylinder given by

$$R = \{x^2 + y^2 \leq 1 \text{ and } -1 \leq z \leq 1\}$$

- We want to compute the outward flux through ∂R of the vector field $\vec{F} = \vec{r}$,

$$\int_{\partial R} \vec{F} \cdot d\vec{S}$$

- This can be computed directly by computing 3 flux integrals, through the top, bottom, and side of the cylinder
- The divergence of $\vec{F}$ is

$$\vec{\nabla} \cdot \vec{F} = \vec{\nabla} \cdot (\vec{i}x + \vec{j}y + \vec{k}z) = 1 + 1 + 1 = 3$$

- By the divergence theorem,

$$\int_{\partial R} \vec{F} \cdot d\vec{S} = \int_{R} \vec{\nabla} \cdot \vec{F}\, dV = 3 \int_{R} dV = 3(\text{volume of } R)$$

- R is a cylinder with radius 1 and height 2 and therefore its volume is $\pi(1)^2(2) = 2\pi$
- Therefore,

$$\int_{\partial R} \vec{F} \cdot d\vec{S} = 6\pi$$

Flux Out Of Cylinder Computed Directly

- Suppose R is the solid circular cylinder given by

$$R = \{x^2 + y^2 \leq 9 \text{ and } 0 \leq z \leq 2\}$$

with boundary ∂R, oriented outward

- We want to compute

$$\int_{\partial R} \vec{F} \cdot d\vec{S},$$

where $\vec{F} = -\vec{i}ye^{x^2+y^2} + \vec{j}xe^{x^2+y^2} + \vec{k}e^{-z^2}$

- The divergence of $\vec{F}$ is a big mess
- On the other hand,

$$\int_{\partial R} \vec{F} \cdot d\vec{S} = \int_{\text{top}} \vec{F} \cdot d\vec{S} + \int_{\text{bottom}} \vec{F} \cdot d\vec{S} + \int_{\text{side}} \vec{F} \cdot d\vec{S}$$

- A normal to the side is $\vec{n} = \vec{i}x + \vec{j}y$ and $\vec{n} \cdot \vec{F} = 0$
- The outward unit normal of the top is $\vec{k}$ and of the bottom is $-\vec{k}$
- Therefore,

$$\begin{aligned}
\int_{\partial R} \vec{F} \cdot d\vec{S} &= \int_{\text{side}} \vec{F} \cdot \vec{n}\, dA + \int_{\text{top}} \vec{F} \cdot \vec{k}\, dA + \int_{\text{bottom}} \vec{F} \cdot (-\vec{k})\, dA \\
&= \int_{\text{top}} e^{-z^2}\, dA - \int_{\text{bottom}} e^{-z^2}\, dA \\
&= e^{-4}(\pi 3^2) - 1(\pi 3^2) = 9\pi(e^{-4} - 1)
\end{aligned}$$

Flux out of a Cube

- Find the outward flux of

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$$

through the boundary of a cube with sides of length ℓ

- The divergence of $\vec{F}$ is

$$\vec{\nabla} \cdot \vec{F} = (F_1)_x + (F_2)_y + (F_3)_z$$

$$F_1 = \frac{x}{(x^2 + y^2 + z^2)^{3/2}} = x(x^2 + y^2 + z^2)^{-3/2}$$

$$(F_1)_x = (x^2 + y^2 + z^2)^{3/2} + x\left(-\frac{3}{2}\right)(x^2 + y^2 + z^2)^{-5/2}(2x)$$

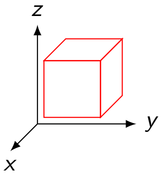
$$= \frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 + z^2)^{5/2}} = \frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\vec{\nabla} \cdot \vec{F} = \frac{(-2x^2 + y^2 + z^2) + (-2y^2 + x^2 + z^2) + (-2z^2 + x^2 + y^2)}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= 0$$

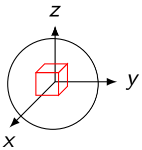
- Use Divergence Theorem?

Flux out of a Cube Not Containing Origin



- Find the outward flux of
$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$$
through the boundary ∂C of a cube C with sides of length ℓ
- $\vec{\nabla} \cdot \vec{F} = 0$
- If the origin lies outside the cube C, then $\vec{F}$ and $\vec{\nabla} \cdot \vec{F}$ are defined on all of C and therefore, by the Divergence Theorem,
$$\int_{\partial C} \vec{F} \cdot d\vec{S} = \int_C \vec{\nabla} \cdot \vec{F} \, dV = 0$$

Flux out of a Cube



- Find the outward flux of $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$ through ∂C
- $\vec{\nabla} \cdot \vec{F} = 0$ everywhere except origin, so the Divergence Theorem cannot be used directly.
- If S is the sphere of radius R centered at the origin, then
$$\int_S \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot \vec{n}\, dA = \int_S \frac{\vec{r}}{|\vec{r}|^3} \cdot \frac{\vec{r}}{|\vec{r}|}\, dA = \frac{1}{R^2} \int_S dA = 4\pi$$
- If R is big enough, then there is a region D whose boundary, oriented outward is $\partial D = S \cup (\partial C)_-$
- Since $\vec{F}$ and $\vec{\nabla} \cdot \vec{F}$ are defined on all of D, by the Divergence Theorem,
$$0 = \int_D \vec{\nabla} \cdot \vec{F}\, dV = \int_{\partial D} \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot d\vec{S} - \int_{\partial C} \vec{F} \cdot d\vec{S}$$
and therefore
$$\int_{\partial C} \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot d\vec{S} = 4\pi$$

More Examples

- $\int_S (\vec{i} + 2\vec{j} - 5\vec{k}) \cdot d\vec{S}$, where S is the sphere of radius 5 centered at $(1, 1, 1)$
- $\int_H (\vec{i} + 2\vec{j} - 5\vec{k}) \cdot d\vec{S}$, where H is the upper half of the sphere of radius 5 centered at $(1, 1, 1)$

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