MATH-UA 123 Calculus 3: Flux Integrals, Stokes' Theorem

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Flux Integral

- Consider:
  - Oriented surface S
  - Vector field $\vec{F}$ along S
- The flux integral of $\vec{F}$ through S can be written as

$$\int_S \vec{F} \cdot d\vec{S}$$

- There are two ways to calculate a flux integral
  - If $\vec{n}$ is the positively oriented unit normal to S, and $\vec{F} \cdot \vec{n}$ is constant on S, then flux integral is given by

$$\int_S \vec{F} \cdot d\vec{S} = (\vec{F} \cdot \vec{n})(\text{Area}(S)).$$

  - If $\vec{r} : D \to S$, where D is a domain in $\mathbb{R}^2$, such that $\vec{r}_s \times \vec{r}_t(s,t)$ is a positively oriented normal, then the flux integral is given by

$$\int_S \vec{F} \cdot d\vec{S} = \int_D \vec{F}(\vec{r}(s,t)) \cdot (\vec{r}_s \times \vec{r}_t(s,t)\, ds\, dt$$

Example

- Consider
$$\int_S (\vec{i}zx + \vec{j}zy) \cdot d\vec{S},$$
where S is the upper hemisphere of radius R centered at the origin oriented outward
- The positively oriented unit normal of S at $\vec{r} \in S$ is
$$\vec{n} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{i}x + \vec{j}y + \vec{k}z}{R}$$
- Here, $\vec{F} = \vec{i}x + \vec{j}y$ and therefore
$$\vec{F} \cdot \vec{n} = (\vec{i}zx + \vec{j}zy) \cdot \left(\frac{\vec{i}x + \vec{j}y + \vec{k}z}{R} \right) = \frac{z(x^2 + y^2)}{R},$$
which is not constant on S
- Therefore, we must parameterize S

Example: Parameterization of Sphere

- Use spherical coordinates to parameterize S
- Given

$$0 \leq \phi \leq \frac{\pi}{2} \text{ and } 0 \leq \theta \leq 2\pi,$$

$$\begin{aligned}
\vec{r}(\phi,\theta) &= R(\vec{i}\sin\phi\cos\theta + \vec{j}\sin\phi\sin\theta + \vec{k}\cos\phi) \\
\vec{r}_\phi &= R(\vec{i}\cos\phi\cos\theta + \vec{j}\cos\phi\sin\theta - \vec{k}\sin\phi) \\
\vec{r}_\theta &= R(-\vec{i}\sin\phi\sin\theta + \vec{j}\sin\phi\cos\theta) \\
\vec{r}_\phi \times \vec{r}_\theta &= R(\vec{i}\cos\phi\cos\theta + \vec{j}\cos\phi\sin\theta - \vec{k}\sin\phi) \times R(-\vec{i}\sin\phi\sin\theta + \vec{j}\sin\phi\cos\theta \\
&= R^2(\vec{i}(\sin\phi)^2\cos\theta + \vec{j}(\sin\phi)^2\sin\theta + \vec{k}(\cos\phi\sin\phi)((\cos\theta)^2 + (\sin\theta)^2)) \\
&= R^2(\sin\phi)(\vec{i}\sin\phi\cos\theta + \vec{j}\sin\phi\sin\theta + \vec{k}\cos\phi) \\
&= R^2(\sin\phi)\frac{\vec{r}}{|\vec{R}|} \\
&= R(\sin\phi)(\vec{i}x + \vec{j}y + \vec{k}z)
\end{aligned}$$

- Since $R^2(\sin\phi) > 0$ when $0 < \phi < \pi$, $\vec{r}_\phi \times \vec{r}_\theta$ is pointing outward and therefore has the correct orientation
- The vector field is

$$\vec{F} = \vec{i}zx + \vec{j}zy = R^2\cos\phi\sin\phi(\vec{i}\cos\theta + \vec{j}\sin\theta)$$

Computation of Flux Integal Using Parameterization

Using the formulas we have,

$$
\begin{aligned}
\int_S \vec{F} \cdot d\vec{S} &= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta)\, d\theta\, d\phi \\
&= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} (R\cos\phi)(\vec{i}x + \vec{j}y) \cdot R(\sin\phi)(\vec{i}x + \vec{j}y + \vec{k}z)\, d\phi\, d\theta \\
&= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} R^2(\cos\phi \sin\phi)(x^2 + y^2)\, d\phi\, d\theta \\
&= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} R^4(\cos\phi)(\sin\phi)^3\, d\phi\, d\theta \\
&= 2\pi R^4 \int_{u=0}^{u=1} u^3\, du \\
&= \frac{1}{2}\pi R^3
\end{aligned}
$$

Fundamental Theorems of Calculus

- (Fundamental Theorem of Calculus)

$$\int_{t=a}^{t=b} f'(t)\, dt = f(b) - f(a)$$

- (Fundamental Theorem of Line Integrals) Given an oriented curve C from $\vec{r}_{\text{start}}$ to $\vec{r}_{\text{end}}$,

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}_{\text{end}}) - f(\vec{r}_{\text{start}})$$

- (Green's Theorem) Given a domain D in 2-space with positively oriented boundary ∂D:

$$\int_D \vec{\nabla} \times \vec{F}\, dA = \int_{\partial D} \vec{F} \cdot d\vec{r}$$

- (Stokes' Theorem) Given an oriented surface S in 3-space with positively oriented boundary ∂S:

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

- (Divergence Theorem) Given a domain R with positively oriented boundary ∂R in 3-space,

$$\int_R \vec{\nabla} \cdot \vec{F}\, dV = \int_{\partial R} \vec{F} \cdot d\vec{S}$$

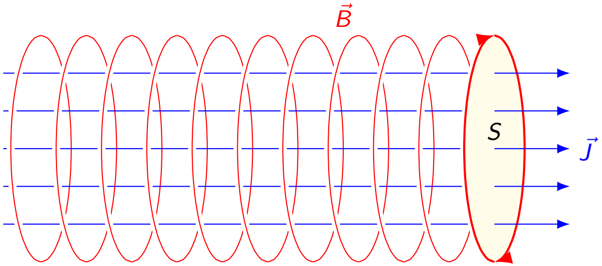
General Form of Each Fundamental Theorem of Calculus

- Always a tradeoff between integrating a derivative or integrating over the boundary

$$\int_{\text{Domain}} \text{Some kind of derivative of a function or vector field}$$
$$= \int_{\text{Boundary of domain}} \text{Function or vector field itself}$$

- Integral is always over an oriented domain or its oriented boundary
- General facts about integrals
  - The integral over a domain must always be an integral of a function
  - The integral over an oriented curve in 2D or 3D space must be an integral of a vector field
  - The integral over an oriented surface in 3-space must be an integral of a vector field

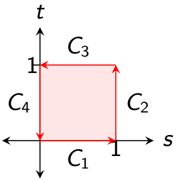
Towards Stokes's Theorem: Ampère's Law

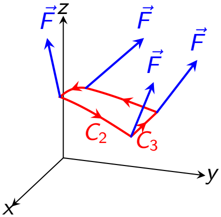


- An electrical current $\vec{J}$ passing through a wire induces a magnetic field $\vec{B}$ around the wire
- Ampère's law:

$$\int_C \vec{B} \cdot d\vec{r} = \int_S \vec{J} \cdot d\vec{S}$$

Line Integral Around Boundary of Surface Parameterized By Rectangle





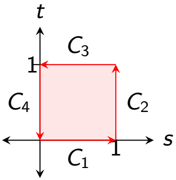
- Suppose $\vec{r}(s,t)$, $0 \leq s, t \leq 1$, is a parameterization of a small surface S
- The boundary of S is $C = C_1 \cup C_2 \cup C_3 \cup C_4$, where the respective parameterizations are

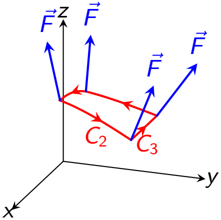
$$
\begin{aligned}
\text{Along } C_1: &\ \ \vec{r}(s) = \vec{r}(s,0),\ d\vec{r} = \vec{r}_s(s,0)\,ds,\ 0 \leq s \leq 1 \\
\text{Along } C_2: &\ \ \vec{r}(t) = \vec{r}(1,t),\ d\vec{r} = \vec{r}_t(1,t)\,dt,\ 0 \leq t \leq 1 \\
\text{Along } C_3: &\ \ \vec{r}(s) = \vec{r}(s,1),\ d\vec{r} = \vec{r}_s(s,1)\,ds,\ 1 \geq s \geq 0 \\
\text{Along } C_4: &\ \ \vec{r}(t) = \vec{r}(0,t),\ d\vec{r} = \vec{r}_t(0,t)\,dt,\ 1 \geq t \geq 0
\end{aligned}
$$

- The line integral of $\vec{F}$ around C is

$$
\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r}
$$

Line Integral Calculation





The line integral of $\vec{F}$ around C is

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r}$$

$$= \int_{s=0}^{s=1} \vec{F}(\vec{r}(s,0))\cdot \vec{r}_s(s,0)\,dt + \int_{s=1}^{s=0} \vec{F}(\vec{r}(s,1))\cdot \vec{r}_s(s,1)\,ds$$

$$+ \int_{t=0}^{t=1} \vec{F}(\vec{r}(1,t))\cdot \vec{r}_t(1,t)\,dt + \int_{t=1}^{t=0} \vec{F}(\vec{r}(0,t))\cdot \vec{r}_t(0,t)\,dt$$

$$= \int_{t=0}^{t=1} \vec{F}(\vec{r}(1,t))\cdot \vec{r}_t(1,t) - \vec{F}(\vec{r}(0,t))\cdot \vec{r}_t(0,t)\,dt$$

$$- \int_{s=0}^{s=1} \vec{F}(\vec{r}(s,1))\cdot \vec{r}_s(s,1) - \vec{F}(\vec{r}(s,0))\cdot \vec{r}_s(s,0)\,ds$$

By the Chain Rule and the Fundamental Theorem of Calculus

$$\begin{aligned}
&\int_{t=0}^{t=1} \vec{F}(\vec{r}(1,t)) \cdot \vec{r}_t(1,t) - \vec{F}(\vec{r}(0,t)) \cdot \vec{r}_t(0,t)\, dt \\
&= \int_{t=0}^{t=1} \int_{s=0}^{s=1} \partial_s(\vec{F}(\vec{r}(s,t)) \cdot \vec{r}_t(s,t))\, ds\, dt \\
&= \int_{t=0}^{t=1} \int_{s=0}^{s=1} (\vec{F}_x x_s + \vec{F}_y y_s + \vec{F}_z z_s) \cdot \vec{r}_t + \vec{F} \cdot \vec{r}_{ts}\, ds\, dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_{s=0}^{s=1} \vec{F}(\vec{r}(s,1)) \cdot \vec{r}_s(s,1) - \vec{F}(\vec{r}(s,0)) \cdot \vec{r}_s(s,0)\, ds \\
&= \int_{s=0}^{s=1} \int_{t=0}^{t=1} \partial_t(\vec{F}(\vec{r}(s,t)) \cdot \vec{r}_s(s,t))\, ds\, dt \\
&= \int_{s=0}^{s=1} \int_{t=0}^{t=1} (\vec{F}_x x_t + \vec{F}_y y_t + \vec{F}_z z_t) \cdot \vec{r}_s + \vec{F} \cdot \vec{r}_{st}\, ds\, dt
\end{aligned}$$

Subtracting

$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{s=0}^{s=1} \int_{t=0}^{t=1} \vec{F}_x \cdot (x_s \vec{r}_t - x_t \vec{r}_s) + \vec{F}_y (y_s \vec{r}_t - y_t \vec{r}_s) + \vec{F}_z (z_s \vec{r}_t - z_t \vec{r}_s)\, ds\, dt$$

$$= \int_{s=0}^{s=1} \int_{t=0}^{t=1} \vec{F}_x \cdot (\vec{j}(x_s y_t - x_t y_s) + \vec{k}(x_s z_t - x_t z_s))$$

$$+ \vec{F}_y \cdot (\vec{i}(y_s x_t - y_t x_s) + \vec{k}(y_s z_t - y_t z_s)) + \vec{F}_z \cdot (\vec{i}(z_s x_t - z_t x_s) + \vec{k}(z_s y_t - z_t y_s))\, ds\, dt$$

$$= \int_{s=0}^{s=1} \int_{t=0}^{t=1} ((F_2)_x - (F_1)_y)(x_s y_t - x_t y_s) + ((F_3)_y - (F_2)_z)(y_s z_t - y_t z_s)$$

$$+ ((F_1)_z - (F_3)_x)(z_s x_t - z_t x_s)\, ds\, dt$$

$$= \int_{s=0}^{s=1} \int_{t=0}^{t=1} (\vec{i}((F_3)_y - (F_2)_z) + \vec{j}((F_1)_z - (F_3)_x) + \vec{k}((F_2)_x - (F_1)_y))$$

$$\cdot (\vec{i}(y_s z_t - y_t z_s) + \vec{j}(z_s x_t - z_t x_s) + \vec{k}(x_s y_t - x_t y_s))\, ds\, dt$$

$$= \int_{s=0}^{s=1} \int_{t=0}^{t=1} (\vec{\nabla} \times \vec{F}) \cdot (\vec{r}_s \times \vec{r}_t)\, dt\, ds = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

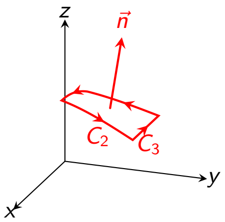
Stokes' Theorem For Surface Parameterized By Rectangle

- Suppose S is a surface in a 3D region R with a parameterization $\vec{r}(s,t)$, $(s,t) \in [0,1] \times [0,1]$
- Given a vector field $\vec{F}$ on D,

$$\int_C \vec{F} \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S},$$

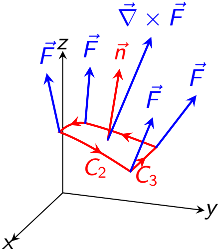
where the orientation on S is given by $\vec{r}_s \times \vec{r}_t$ and C is the boundary of S with the orientation corresponding to going counterclockwise around the boundary of the rectangle $[0,1] \times [0,1]$

Orientation of Surface With Boundary in 3-Space



- Suppose S is an oriented surface in 3-space with boundary C
- An orientation of the curve C is positive with respect to the orientation of S, if it satisfies the righthand rule.
  - If you walk along the curve with your head pointing in the direction of the oriented normal to S, then the surface is on your left
  - If you point the fingers of your right hand in the direction of the orientation of the curve and curl them towards the surface, then your thumb points in the direction of the oriented normal to the surface
  - If you point the thumb of your right hand in the direction of the normal to S, your fingers should point in the direction of the orientation of C

Stokes' Theorem for a Bounded Surface



- Suppose S is an oriented surface with positively oriented boundary C
- If we chop up $S = S_1 \cup \cdots \cup S_N$, where
  - Each S_k can be parameterized by a rectangle
  - The intersection of any two S_j and S_k is either empty or a side of each rectangle,

then

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_{C_1} \vec{F} \cdot d\vec{r} + \cdots + \int_{C_N} \vec{F} \cdot d\vec{r} \\ &= \int_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} + \cdots + \int_{S_N} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} \\ &= \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}\end{aligned}$$

Stokes' Theorem for a Bounded Surface

Theorem

Suppose S is an oriented surface inside a 3D region R, whose boundary is C with the positive orientation. If $\vec{F}$ is a vector field on R, then

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

In particular, if S is a closed oriented surface (i.e., it has no boundary), then

$$\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0$$

Corollary

If S_1 and S_2 are any two oriented surfaces that have the same positively oriented boundary (i.e., $\partial S_1 = \partial S_2$), then for any vector field $\vec{F}$,

$$\int_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}.$$

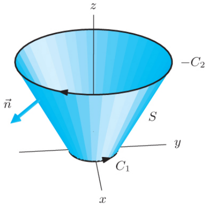
Example of Line Integral Around Circle

- Want to compute
$$\int_C \vec{F} \cdot d\vec{r},$$
where C is the circle of radius 3 in the plane $z = 2$ and centered around the z-axis, and
$$\vec{F} = \vec{i}y + \vec{j}2x + \vec{k}z^2$$
- Could do this directly:
$$\begin{aligned}
\vec{r}(\theta) &= \langle 3\cos\theta, 3\sin\theta, 2\rangle \\
d\vec{r} &= \vec{r}'(\theta)\, d\theta \\
&= \langle -3\sin\theta, 3\cos\theta, 0\rangle\, d\theta \\
\vec{F} &= \langle 3\sin\theta, 6\cos\theta, 9(2^2)\rangle \\
\int_C \vec{F} \cdot d\vec{r} &= \int_{\theta=0}^{\theta=2\pi} -9(\sin\theta)^2 + 18(\cos\theta)^2\, d\theta
\end{aligned}$$
- Need half-angle identities to calculate integral

Use Stokes' Theorem to Compute Line Integral

- Want to compute
$$\int_C \vec{F}\cdot d\vec{r},$$
where C is the circle of radius 3 in the plane $z=2$, centered around the z-axis, oriented counterclockwise, and
$$\vec{F}=\vec{i}y+\vec{j}2x+\vec{k}z^2$$
- Try using Stokes' Theorem:
$$\int_C \vec{F}\cdot d\vec{r}=\int_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$
- Calculate curl of $\vec{F}$:
$\vec{\nabla}\times\vec{F}=\vec{i}((F_3)_y-(F_2)_z)+\vec{j}((F_1)_z-(F_3)_x)+\vec{k}((F_2)_x-(F_1)_y)=\vec{k}(2-1)=\vec{k}$
- Can choose any oriented surface S whose positively oriented boundary is C
- Simplest choice is disk in the plane $z=3$
$$S=\{x^2+y^2\leq 9 \text{ and } z=2\}$$
- By Stokes' Theorem,
$$\int_C \vec{F}\cdot d\vec{r}=\int_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S}=\int_S(\vec{F}\cdot\vec{n})\,dA$$
$$=\int_S\vec{k}\cdot\vec{k}\,dA=\int_S dA=9\pi$$

Application of Stokes' Theorem



- Suppose $C_2 = \{x^2 + 9y^2 = 9 \text{ and } z = 5\}$ and we want to compute

$$\int_{C_2} \left(\frac{-\vec{i}y + \vec{j}x}{x^2 + y^2} \right) \cdot d\vec{r}$$

- Computing this directly will be a mess
- But recall that the vector field $\vec{F} = \dfrac{-\vec{i}y + \vec{j}x}{x^2 + y^2}$ satisfies $\vec{\nabla} \times \vec{F} = 0$
- Also, if C_1 is the unit circle in the xy-plane centered at the origin and oriented counterclockwise, then the line integral of $\vec{F}$ around C_1 is easy to calculate
- So let S be a surface whose boundary is $C_1 \cup C_2$
- Orient S using an outward normal
- Orient C_1 and C_2 in the counterclockwise direction
- The positive orientation for ∂S is $C_1 \cup (-C_2)$