MATH-UA 123 Calculus 3: Parametric Surfaces, Flux Integrals

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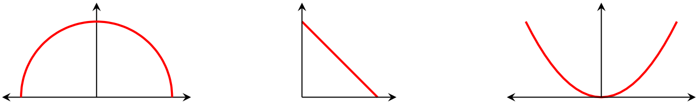
START RECORDING

Two Ways to Describe a Curve in 2-Space



As a contour:

$$x^2 + y^2 = 1, \quad -1 \le x \le 1,\ \ y \ge 0$$

$$x + y = 1, \quad 0 \le x, y \le 1$$

$$y - x^2 = 0, \quad 0 \le x, y \le 1$$

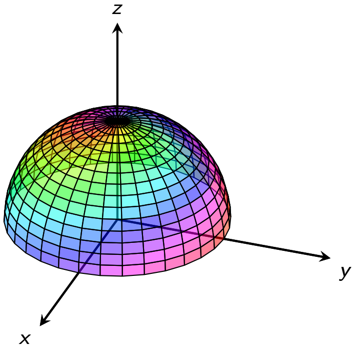
As a parameterized curve:

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, \quad 0 \le t \le \pi$$

$$\vec{r}(x) = \langle x, 1 - x \rangle, \quad 0 \le x \le 1$$

$$\vec{r}(x) = \langle x, x^2 \rangle, \quad 0 \le x \le 1$$

Two Ways to Describe a Surface



- As a contour:

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ z &\geq 0 \end{aligned}$$

- As a parameteric surface:

$$\vec{r}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$$

$$0 \leq \phi \leq \frac{\pi}{2}, \ 0 \leq \theta \leq 2\pi$$

Two Ways to Describe a Surface

- A contour of a function $f(x, y, z)$ on 3-space
  - Suppose $f(x, y, z)$ is a function on a domain D in 3-space
  - The contour $f = c$ is the set

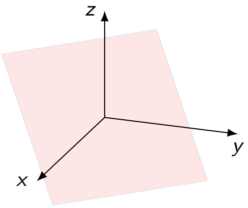
$$S = \{(x, y, z) \in D \ : \ f(x, y, z) = c\}$$

  - If, for every $(x, y, z) \in S$, $\vec{\nabla} f(x, y, z) \neq 0$, then S is a surface
- A parametric surface
  - Parametric domain D in 2-space
  - A set S in 3-space
  - A map

$$\begin{aligned} \vec{r} &: D \to S \\ (s, t) &\mapsto \vec{r}(s, t) = \vec{i}x(s, t) + \vec{j}y(s, t) + \vec{k}z(s, t), \end{aligned}$$

  - If, for each $(s, t) \in D$, $\vec{r}_s(s, t) \times \vec{r}_t(s, t) \neq 0$, then S is a surface

Example: Plane



- Contour:

$$ax + by + cz = d,$$

  where at least one of a, b, c is nonzero
- Parameteric surface: If $c \neq 0$, then the plane can be parameterized by

$$\vec{r}(s,t) = \left\langle s, t, \frac{d - as - bt}{c} \right\rangle, \quad -\infty < s, t < \infty$$

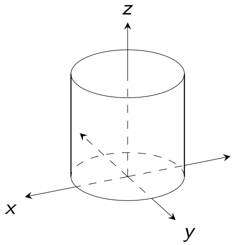
- Check parameterization:

$$\vec{r}_s = \left\langle 1, 0, \frac{-a}{c} \right\rangle$$

$$\vec{r}_t = \left\langle 0, 1, -\frac{b}{c} \right\rangle$$

$$\vec{r}_s \times \vec{r}_t = \left\langle \frac{a}{c}, \frac{b}{c}, 1 \right\rangle$$

Example: Cylinder Side



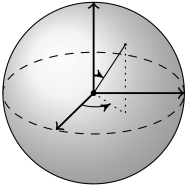
- Contour: $x^2 + y^2 = R^2$ and $0 \le z \le h$
- Parametric surface:

$$\vec{r}(s,t) = \vec{i}R\cos s + \vec{j}R\sin s + \vec{k}t,\ 0 \le s \le 2\pi \text{ and } 0 \le t \le h,$$

- Where

$$\begin{aligned}
\vec{r}_s &= R(-\vec{i}\sin s + \vec{j}\cos s) \\
\vec{r}_t &= \vec{k} \\
\vec{r}_s \times \vec{r}_t &= R(-\vec{i}\sin s + \vec{j}\cos s) \times \vec{k} \\
&= R(\vec{i}\cos s + \vec{j}\sin s)
\end{aligned}$$

Example: Sphere



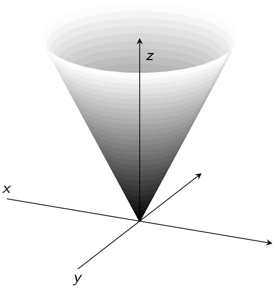
- Contour: $x^2 + y^2 + z^2 = R^2$, where $R > 0$
- Parametric surface:

$$\vec{r}(\phi, \theta) = \vec{i}(R \sin\phi \cos\theta) + \vec{j}(R \sin\phi \sin\theta) + \vec{k}(R \cos\phi),\ 0 \le \pi \le \pi \text{ and } 0 \le \theta \le 2\pi,$$

where

$$\begin{aligned}
\vec{r}_\phi &= R(\vec{i} \cos\phi \cos\theta + \vec{j} \cos\phi \sin\theta - \vec{k} \sin\phi) \\
\vec{r}_\theta &= R(-\vec{i} \sin\phi \sin\theta + \vec{j} \sin\phi \cos\theta) \\
\vec{r}_\phi \times \vec{r}_\theta &= R^2(\vec{i}(\sin\phi)^2 \cos\theta + \vec{j}(\sin\phi)^2 \sin\theta + \vec{k} \sin\phi \cos\phi) \\
&= (R \sin\phi)(\vec{i}(R \sin\phi \cos\theta) + \vec{j}(R \sin\phi \sin\theta) + \vec{k}(R \cos\phi)) \\
&= (R \sin\phi)\vec{r}(\phi, \theta) \\
&\ne \vec{0} \text{ if and only if } \phi = 0 \text{ or } \pi
\end{aligned}$$

Example: Cone



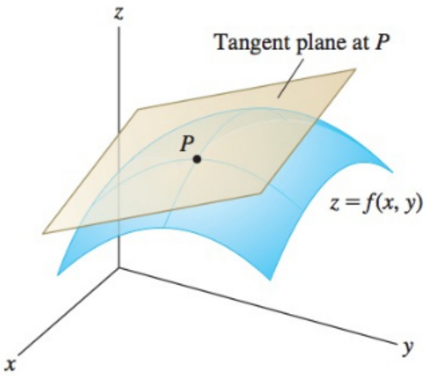
- Contour: $m^2(x^2+y^2)-z^2=0$, where $z \geq 0$ and $m > 0$
- Parameteric surface:

$$\vec{r}(\theta, r) = \vec{i}(t\cos\theta) + \vec{j}(t\sin\theta) + \vec{k}(mt),\ 0 \leq \theta \leq 2\pi,\ -\infty < t < \infty,$$

  where

$$\begin{aligned}
\vec{r}_\theta &= -i(t\sin\theta) + \vec{j}(t\cos\theta) \\
\vec{r}_t &= \vec{i}\cos\theta + \vec{j}\sin\theta + \vec{k}m \\
\vec{r}_\theta \times \vec{r}_t &= \vec{i}(mt\cos\theta) + \vec{j}(mt\sin\theta) - \vec{k}t \\
&= t(\vec{i}(m\cos\theta) + \vec{j}(m\sin\theta) - \vec{k}) \\
&\neq \vec{0} \text{ if } t \neq 0
\end{aligned}$$

Tangent plane at a Point on a Parametric Surface



- Suppose $\vec{r}(u,v) = \vec{i}x(u,v) + \vec{j}y(u,v) + \vec{k}z(u,v)$ is a parameterization of a surface
- At a point $\vec{r}(a,b)$ on the surface, the vectors $\vec{r}_u(a,b)$ and $\vec{r}_v(a,b)$ are tangent to the surface
- If $\vec{r}_u(a,b) \times \vec{r}_v(a,b) \neq \vec{0}$, then the two tangent vectors lie in a plane with normal vector $\vec{r}_u(a,b) \times \vec{r}_v(a,b)$

Example: Circular Paraboloid

- Consider the paraboloid $z = x^2 + y^2$, which has a parameterization

$$\vec{r}(x,y) = \vec{i}x + \vec{j}y + \vec{k}(x^2 + y^2), \ -\infty < x, y < \infty$$

- At the the point $\vec{r}(0,0) = \vec{i}0 + \vec{j}0 + \vec{k}0$, the vectors

$$\vec{r}_x(0,0) = \vec{i}, \ \vec{r}_y(0,0) = \vec{j},$$

  are tangent to the surface and a normal to the surface is

$$\vec{r}_x(0,0) \times \vec{r}_y(0,0) = \vec{k}$$

- At $\vec{r}(\sqrt{3},1) = (\sqrt{3},1,4)$, the vectors

$$\vec{r}_x(\sqrt{3},1) = \vec{i} + \vec{k}(2\sqrt{3}), \ \vec{r}_y(\sqrt{3},1) = \vec{j} + \vec{k}(2),$$

  are tangent to the surface and a normal to the surface is

$$\vec{r}_x \times \vec{r}_y = -\vec{i}2\sqrt{3} - \vec{j}2 + \vec{k}$$

- In general, at a point $\vec{r}(x,y) = \langle x, y, x^2 + y^2 \rangle$, the vectors

$$\vec{r}_x = \vec{i} + \vec{k}2x \text{ and } \vec{r}_y(x,y) = \vec{j} + \vec{k}2y$$

  are tangent to the surface and a normal to the surface is

$$\vec{r}_x \times \vec{r}_y = (\vec{i} + \vec{k}2x) \times (\vec{j} + \vec{k}2y) = -\vec{i}2x - \vec{j}2y + \vec{k},$$

Example: Circular Paraboloid

- Another parameterization of $z = x^2 + y^2$, using cylindrical coordinates, is

$$\vec{r}(r,\theta) = \vec{i}r\cos\theta + \vec{j}r\sin\theta + \vec{k}r^2, \ \ r \geq 0 \text{ and } 0 \leq \theta \leq 2\pi$$

- At each point $\vec{r}(r,\theta)$ on the surface, the vectors

$$\vec{r}_r = \vec{i}\cos\theta + \vec{j}\sin\theta + \vec{k}2r \text{ and } \vec{r}_\theta = -\vec{i}r\sin\theta + \vec{j}r\cos\theta$$

are tangent to the surface and a normal is

$$\begin{aligned} \vec{r}_r \times \vec{r}_\theta &= -\vec{i}2r^2\cos\theta - \vec{j}2r^2\sin\theta + \vec{k}r \\ &= r(-\vec{i}2r\cos\theta - \vec{j}2r\sin\theta + \vec{k}) \\ &= r(-i2x - \vec{j}2y + \vec{k}) \\ &\neq \vec{0} \text{ if and only if } r = 0 \end{aligned}$$

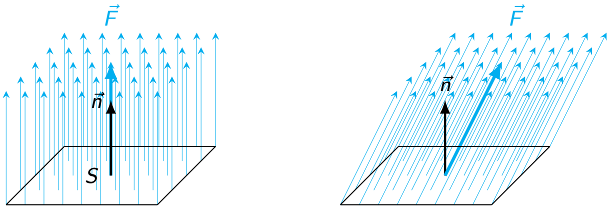
- A unit normal at $\vec{r}(r,\theta)$ is

$$\vec{n} = \frac{\vec{r}_r \times \vec{r}_\theta}{|\vec{r}_r \times \vec{r}_\theta|} = \frac{-i2x - \vec{j}2y + \vec{k}}{\sqrt{1+4r^2}}$$

- At $\vec{r}(0,0) = \vec{i}0 + \vec{j}0 + \vec{k}0$ and $\vec{n} = \vec{k}$
- At $\vec{r}(2, \frac{\pi}{6}) = \vec{i}\sqrt{3} + \vec{j} + \vec{k}4$ and

$$\vec{n} = \frac{-\vec{i}2\sqrt{3} - \vec{j}2 - \vec{k}}{\sqrt{17}}$$

Flux of Constant Vector Field Through Flat Surface



- Flux is the net force of a force field $\vec{F}$ acting on a surface S
- If S is a flat surface and $\vec{F}$ is constant and normal to S, then the net force is

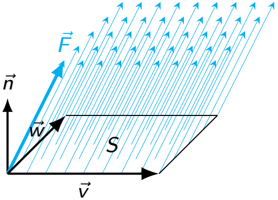
$$\Phi = |\vec{F}|A,$$

  where A is the area of S
- If S is a flat surface and $\vec{F}$ is constant but not necessarily normal to S, then the net force is

$$\Phi = |F|A\cos\theta = (\vec{F} \cdot \vec{n})A,$$

  where $\vec{n}$ is the unit normal to S and θ is the angle between $\vec{F}$ and $\vec{n}$
- **IMPORTANT:** The sign of the flux depends on which unit normal is used
- The choice of which normal to use is called an orientation of S
- The orientation shown can be called the *upward orientation*

Flux of Constant Vector Field Across Parallelogram Using the Cross Product



- If $\vec{v} \times \vec{w}$ has the correct orientation, then let

$$\vec{n} = \frac{\vec{v} \times \vec{w}}{|\vec{v} \times \vec{w}|},$$

- The area of S is

$$A = |\vec{v} \times \vec{w}|$$

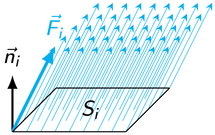
- If $\vec{v} \times \vec{w}$ is the desired orientation, the net flux is

$$\begin{aligned} \Phi &= (\vec{F} \cdot \vec{n})A \\ &= \vec{F} \cdot \left(\frac{\vec{v} \times \vec{w}}{|\vec{v} \times \vec{w}|} \right) |\vec{v} \times \vec{w}| \\ &= \vec{F} \cdot (\vec{v} \times \vec{w}) \end{aligned}$$

- If $\vec{v} \times \vec{w}$ is the wrong orientation, then $\vec{w} \times \vec{v}$ is the orientation and the net flux is

$$\Phi = \vec{F} \cdot (\vec{w} \times \vec{v}) = -\vec{F} \cdot (\vec{v} \times \vec{w})$$

Idea of a Flux Integral



- Suppose surface S is not flat and $\vec{F}$ is not constant
- Use calculus
  - Chop the surface S into small pieces,
  $$S = S_1 \cup \cdots \cup S_N$$
  - Estimate the flux on each small piece S_i:
  $$\Phi_i = \vec{F}_i \cdot \vec{n}_i \, A_i ,$$
  where A_i is the area of S_i
  - Add up the fluxes of the small pieces to get an estimate of the flux across S
  $$\begin{aligned}\Phi &\simeq \Phi_1 + \cdots + \Phi_N \\ &\simeq \Sigma_{i=1}^{N} \vec{F}_i \cdot \vec{n}_i \, A_i\end{aligned}$$
  - Chop S into smaller and smaller pieces and take a limit to get an integral that we write as:
  $$\Phi = \int_S \vec{F} \cdot \vec{n} \, dA = \int_S \vec{F} \cdot d\vec{S},$$
  where we write $d\vec{S} = \vec{n} \, dA$ and sometimes write $dS = dA$
- This is called a flux integral

Calculating a Flux Integral

- **REMEMBER:** If the vector field $\vec{F}$ is constant and S is flat (lies in a plane), then the flux integral is easy to calculation:

$$\int_S \vec{F} \cdot d\vec{S} = (\vec{F} \cdot \vec{n})A,$$

  where $\vec{n}$ is the properly oriented unit normal of S and A is the area of S
- **ANOTHER EASY CASE:** If
  - $\vec{n}$ is a properly oriented unit normal vector field along S
  - $\vec{F} \cdot \vec{n}$ is **CONSTANT** on S (even though S might be curved and $\vec{F}$ might be nonconstant)

  then

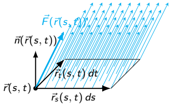
$$\int_S \vec{F} \cdot d\vec{S} = \int_S (\vec{F} \cdot \vec{n})\, dA = (\vec{F} \cdot \vec{n})A,$$

  where A is the area of S
- **NO INTEGRATION NEEDED IN THESE TWO CASES**

Example: Flux of Radial Vector Field Through Sphere

- Suppose S is the sphere of radius R centered at the origin with the outward orientation and
$$\vec{F}(x, y, z) = \vec{i}x + \vec{j}y + \vec{k}z = \vec{r}$$
where p is a scalar constant
- S is given by $x^2 + y^2 + z^2 = R^2$ or, equivalently, $|\vec{r}| = R$
- The position vector $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$ is normal to S at every point on S and points outward
- The outward unit normal at each point $\vec{r}$ on S is therefore
$$\vec{n}(\vec{r}) = \frac{\vec{r}}{|\vec{r}|}$$
- Therefore, at each point on S,
$$\vec{F} \cdot \vec{n} = \vec{r} \cdot \frac{\vec{r}}{R} = \frac{|\vec{r}|^2}{R} = \frac{R^2}{R} = R$$
- The outward flux of $\vec{F}$ through S is therefore
$$\int_S \vec{F} \cdot d\vec{S} = \int_S (\vec{F} \cdot \vec{n})\, dA = (\vec{F} \cdot \vec{n}) \int_S dA = R(4\pi R^2) = 4\pi R^3$$

Calculating a Flux Integral



- Suppose we want to compute a flux integral $\int_S \vec{F} \cdot d\vec{S}$
- Start with a parameterization of S: $\vec{r}(s,t)$, where $(s,t) \in D$
- At each point $\vec{r}(s,t)$ on the surface, the vectors $\vec{r}_s(s,t)\,ds$ and $\vec{r}_t\,dt$ span a small parallelogram tangent to S with area

$$dA = |(\vec{r}_s(s,t)\,ds) \times (\vec{r}_t(s,t)\,dt)| = |\vec{r}_s(s,t) \times \vec{r}_t(s,t)|\,ds\,dt$$

and unit normal

$$\vec{n}(\vec{r}(s,t)) = \frac{(\vec{r}_s(s,t)\,ds) \times (\vec{r}_t(s,t)\,dt)}{|(\vec{r}_s(s,t)\,ds) \times (\vec{r}_t(s,t)\,dt)|} = \frac{\vec{r}_s(s,t) \times \vec{r}_t(s,t)}{|\vec{r}_s(s,t) \times \vec{r}_t(s,t)|}$$

- It follows that

$$\begin{aligned} d\vec{S} &= \vec{F} \cdot \vec{n}\,dA \\ &= \vec{F} \cdot \left(\frac{(\vec{r}_s(s,t)\,ds) \times (\vec{r}_t(s,t)\,dt)}{|(\vec{r}_s(s,t)\,ds) \times (\vec{r}_t(s,t)\,dt)|} \right) |(\vec{r}_s(s,t)\,ds) \times (\vec{r}_t(s,t)\,dt)| \\ &= |\vec{r}_s(s,t) \times \vec{r}_t(s,t)|\,ds\,dt \\ &= \vec{F} \cdot (\vec{r}_s \times \vec{r}_t)\,ds\,dt \end{aligned}$$

Calculating a Flux Integral

- Suppose we want to compute a flux integral $\int_S \vec{F} \cdot d\vec{S}$
- Start with a parameterization of S: $\vec{r}(s,t)$, where $(s,t) \in D$
- We found that

$$\begin{aligned} d\vec{S} &= \vec{F} \cdot \vec{n}\, dA \\ &= \vec{F} \cdot (\vec{r}_s \times \vec{r}_t)\, ds\, dt \end{aligned}$$

- Assming that $\vec{r}_s \times \vec{r}_t$ is the correct orientation, the flux integral can therefore be calculated as follows:

$$\int_S \vec{F} \cdot d\vec{S} = \int_D \vec{F} \cdot (\vec{r}_s \times \vec{r}_t)\, ds\, dt$$

- The integral on the right is a double integral over the 2-dimensional domain D
- It can be calculated using the techniques we learned earlier
- If $\vec{r}_s \times \vec{r}_t$ is the wrong orientation, multiply by -1

Example of Flux Integral

- Suppose S is the graph of $z = 1 - x + 2y$ over the unit disk $x^2 + y^2 \leq 1$, oriented upward, and we want to calculate
$$\int_S z\vec{k} \cdot d\vec{S}$$
- First, check if this is an easy case:
- Since S lies in the plane $x - 2y + z = 1$, a normal vector is $\vec{i} - 2\vec{j} + \vec{k}$ and the corresponding unit normal is
$$\vec{n} = \frac{\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{6}}$$
- Since the coefficient of $\vec{k}$ is positive, it points upward and has the correct orientation
- $\vec{F} \cdot \vec{n} = (z\vec{k}) \cdot \frac{\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{6}} = \frac{z}{\sqrt{6}}$ is not constant
- Not an easy case

Calculate Example Using Parameterization

- Parameterize S: $\vec{r}(x,y) = \vec{i}x + \vec{j}y + \vec{k}(1 - x + 2y)$, where $x^2 + y^2 \leq 1$
- $\vec{r}_x = \vec{i} - \vec{k}$, $\vec{r}_y = \vec{j} + 2\vec{k}$, and therefore

  $$\vec{r}_x \times \vec{r}_y = (\vec{i} - \vec{k}) \times (\vec{j} + 2\vec{k}) = \vec{i} - 2\vec{j} + \vec{k},$$

  which has correct orientation
- The flux integral of $\vec{F} = z\vec{k}$ through S is therefore

  $$\begin{aligned}\int_S z\vec{k} \cdot d\vec{S} &= \int_D z\vec{k} \cdot (\vec{r}_x \times \vec{r}_y)\, dx\, dy \\ &= \int_D (1 - x - y)\vec{k} \cdot (\vec{i} - 2\vec{j} + \vec{k})\, dx\, dy \\ &= \int_D 1 - x - y\, dx\, dy\end{aligned}$$

  where $D = \{x^2 + y^2 \leq 1\}$
- Switch to polar coordinates

  $$\begin{aligned}\int_S z\vec{k} \cdot d\vec{S} &= \int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} (1 - r\cos\theta - r\sin\theta) r\, d\theta\, dr \\ &= \int_{r=0}^{r=1} r\, dr \\ &= \frac{1}{2}\end{aligned}$$

Flux Integral Through Upper Hemisphere

- Let S be the upper half of a sphere with radius R centered at the origin, oriented downwardh
- Compute $\int_S (\vec{i}zx + \vec{j}zy) \cdot d\vec{S}$
- The oriented unit normal is

$$\vec{n} = -\frac{\vec{i}x + \vec{j}y + \vec{k}z}{\sqrt{x^2 + y^2 + z^2}} = -\frac{\vec{i}x + \vec{j}y + \vec{k}z}{R}$$

- $\vec{F} \cdot \vec{n} = (\vec{i}zx + \vec{j}zy) \cdot \vec{n} = -\frac{z(x^2+y^2)}{R}$ is not constant
- Two possible parameterizations
- As a graph: $\vec{r}(x, y) = \vec{i}x + \vec{j}y + \vec{k}\sqrt{R^2 - x^2 - y^2}$, where $x^2 + y^2 \leq R^2$
- Using spherical coordinates:

$$\vec{r}(\phi, \theta) = R(\vec{i}\sin\phi\cos\theta + \vec{j}\sin\phi\sin\theta + \vec{k}\cos\phi), \text{ where } 0 \leq \phi \leq \frac{\pi}{2} \text{ and } 0 \leq \theta \leq 2\pi$$

Use Spherical Coordinates to Calculate Example

- Let S be the upper half of a sphere with radius R centered at the origin, oriented downward
- Compute $\int_S (\vec{i}zx + \vec{j}zy) \cdot d\vec{S}$
- Parameterization using spherical coordinates:

  $$\vec{r}(\phi, \theta) = R(\vec{i}\sin\phi\cos\theta + \vec{j}\sin\phi\sin\theta + \vec{k}\cos\phi), \text{ where } 0 \le \phi \le \frac{\pi}{2} \text{ and } 0 \le \theta \le 2\pi$$

  and therefore

  $$\begin{aligned}
  \vec{F} &= \vec{i}zx + \vec{j}zy = R^2\sin\phi\cos\phi(\vec{i}\cos\theta + \vec{j}\sin\theta) \\
  \vec{r}_\phi &= R(\vec{i}\cos\phi\cos\theta + \vec{j}\cos\phi\sin\theta - \vec{k}\sin\phi) \\
  \vec{r}_\theta &= R(-\vec{i}\sin\phi\sin\theta + \vec{j}\sin\phi\cos\theta) \\
  \vec{r}_\phi \times \vec{r}_\theta &= R^2(\vec{i}(\sin\phi)(\sin\phi\cos\theta) + \vec{j}(\sin\phi)(\sin\phi\sin\theta) \\
  &\quad + \vec{k}((\cos\phi\cos\theta)(\sin\phi\cos\theta) + (\cos\phi\sin\theta)(\sin\phi\sin\theta)) \\
  &= R^2((\sin\phi)^2(\vec{i}\cos\theta + \vec{j}\sin\theta) + \vec{k}(\cos\phi\sin\phi)) \\
  \vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta) &= R^4(\sin\phi)^3\cos\phi
  \end{aligned}$$

- $\vec{r}_\phi \times \vec{r}_\theta$ has the WRONG orientation

Calculation of Example

- Putting this all together,

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{S} &= -\int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta)\, d\theta\, d\phi \\
&= -\int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} R^4 (\sin\phi)^3 \cos\phi\, d\theta\, d\phi \\
&= -2\pi R^4 \left. \frac{(\sin\phi)^4}{4} \right|_{\phi=0}^{\phi=\frac{\pi}{2}} \\
&= -\frac{\pi R^4}{2}
\end{aligned}$$