MATH-UA 123 Calculus 3: Green's Theorem

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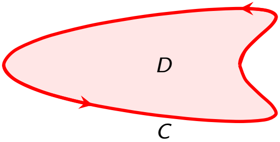
Fundamental Theorem of Line Integrals

- Let C be an oriented closed curve
- Let $\vec{F} = \vec{\nabla} f$ be a conservative vector field
- Then
$$\int_C \vec{F} \cdot d\vec{r} = 0$$
- Equivalently,
$$\int_C f_x \, dx + f_y \, dy = 0$$

Fundamental Theorem of Line Integrals (Special Case)



- Let $D \subset \mathbb{R}^2$ be a simply connected domain
- Let $C = \partial D$ be boundary of D
- If $\vec{F}$ is a vector field on D such that

$$\vec{\nabla} \times \vec{F} = 0,$$

then

$$\int_C \vec{F} \cdot d\vec{r} = 0,$$

- Proof:
  - If D is simply connected and $\vec{\nabla} \times \vec{F} = 0$, then $\vec{F}$ is a gradient field
  - The conclusion now follows by the Fundamental Theorem of Line Integrals

What happens if $\vec{\nabla} \times \vec{F} \neq 0$?

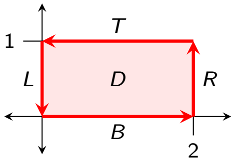
- Suppose
  - D is a simply connected domain
  - $C = \partial D$ is the boundary of D oriented counterclockwise
  - $\vec{F}$ is a vector field on D
- Green's Theorem says

$$\int_C \vec{F} \cdot d\vec{r} = \int_D \vec{\nabla} \times \vec{F} \, dA$$

- Equivalently, if $\vec{F} = \vec{i}P + \vec{j}Q$, then

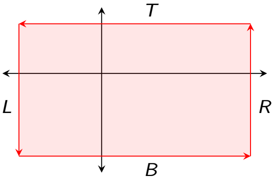
$$\int_C P\,dx + Q\,dy = \int_C Q_x - P_y \, dA$$

Example



- Consider
$$\int_C -y\,dx + x\,dy,$$
where $C = B \cup R \cup T \cup L$
- On one hand,
$$\begin{aligned}
&\int_C -y\,dx + x\,dy \\
&= \int_B -y\,dx + x\,dy + \int_R -y\,dx + x\,dy + \int_T -y\,dx + x\,dy + \int_L -y\,dx + x\,dy \\
&= \int_{x=0}^{x=2} -0\,dx + \int_{y=0}^{y=1} 2\,dy + \int_{x=2}^{x=0} -1\,dx + \int_{y=1}^{y=0} 0\,dy \\
&= 2 + 2 = 4
\end{aligned}$$
- On the other hand,
$$\int_D \partial_x(x) - \partial_y(-y)\,dA = \int_D 2\,dA = 2(\text{area of } D) = 4$$

Green's Theorem on a Rectangle



- Consider the domain $D = [a, b] \times [c, d]$
- Let C = boundary of $D = R \cup T \cup L \cup B$, oriented counterclockwise
- By the Fundamental Theorem of Calculus,

$$\int_C P\,dx + Q\,dy$$

$$= \int_B P\,dx + Q\,dy + \int_R P\,dx + Q\,dy + \int_T P\,dx + Q\,dy + \int_L P\,dx + Q\,dy$$

$$= \int_{x=a}^{x=b} P(x,c)\,dx + \int_{y=c}^{y=d} Q(b,y)\,dy - \int_{x=a}^{x=b} P(x,d)\,dx - \int_{y=c}^{y=d} Q(a,y)\,dy$$

$$= \int_{x=a}^{x=b} P(x,c) - P(x,d)\,dx + \int_{y=c}^{y=d} Q(b,y) - Q(a,y)\,dy$$

$$= \int_{x=a}^{x=b} \int_{y=c}^{y=d} -P_y(x,y)\,dy\,dx + \int_{y=c}^{y=d} \int_{x=a}^{x=b} Q_x(x,y)\,dx\,dy$$

$$= \int_D Q_x - P_y\,dA$$

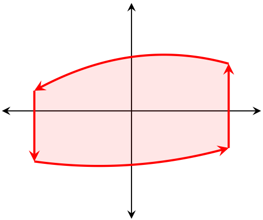
Green's Theorem on a Rectangle

Theorem

If D is a rectangle, C is the boundary of D oriented counterclockwise, and $\vec{F} = \vec{i}P + \vec{j}Q$ is a vector field on D,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy = \int_D Q_x - P_y\,dA = \int_D \vec{\nabla} \times \vec{F}\,dA.$$

Green's Theorem on a Simply Connected Domain



Green's Theorem can be generalized to a simply connected domain:

Theorem

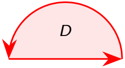
If D is a simply connected domain with boundary ∂D oriented counterclockwise, then, for any vector field $\vec{F}$ on D,

$$\int_C \vec{F} \cdot d\vec{r} = \int_D \vec{\nabla} \times \vec{F} \, dA.$$

Equivalently, given any two functions P and Q on D,

$$\int_C P\,dx + Q\,dy = \int_D Q_x - P_y \, dA.$$

Example



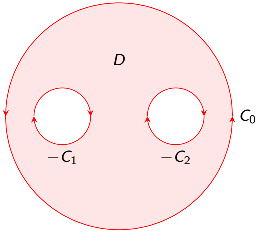
- Let D be the upper half of a circular disk of radius R centered at the origin
- Consider the line integral

$$\int_{\partial D} (x^2 + y^2)\, dx + 2xy\, dy$$

- By Green's Theorem,

$$\begin{aligned}
\int_{\partial D} (x^2 + y^2)\, dx - 2xy\, dy &= \int_D (-2xy)_x - (x^2 + y^2)_y\, dA \\
&= \int_D -2y - 2y\, dA \\
&= -4 \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=\pi} r \sin\theta\, r\, d\theta\, dr \\
&= -4 \int_{r=0}^{r=R} r^2\, dr \int_{\theta=0}^{\theta=\pi} \sin\theta\, r\, d\theta \\
&= -4 \left(\frac{R^3}{3}\right) \left(-\cos\theta|_{\theta=0}^{\theta=\pi}\right) \\
&= \frac{8}{3}
\end{aligned}$$

Green's Theorem on a Non-Simply Connected Domain



- Let C_0 be a connected closed curve in 2-space, oriented counterclockwise
- Let $C_1, \dots, C_N$ be connected closed curves, oriented counterclockwise, inside C
- For each k, let $-C_k$ be the curve C_k but with the opposite orientation (clockwise)
- Let D be the domain that lies inside C but outside the curves $C_1, \dots, C_N$

Theorem

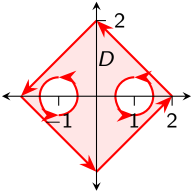
Given $C = C_0 \cup (-C_1) \cup \cdots \cup (-C_N)$ and a vector field $\vec{F} = \vec{i}P + \vec{j}Q$ on D,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_{C_0} \vec{F} \cdot d\vec{r} - \int_{C_1} \vec{F} \cdot d\vec{r} - \cdots - \int_{C_N} \vec{F} \cdot d\vec{r} \\ &= \int_D Q_x - P_y \, dA.\end{aligned}$$

Equivalently, given functions P and Q,

$$\int_C P\,dx + Q\,dy = \int_D Q_x - P_y \, dA.$$

Example



- Let C_1 and C_2 be the two circles of radius $\frac{1}{2}$ shown above and C_3 the outer boundary of D, all oriented counterclockwise
- Let D be the domain inside C_3 but outside C_1 and C_2
- Suppose we want to calculate

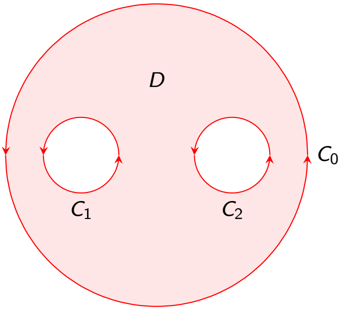
$$\int_{\partial D} -y\,dx + x\,dy,$$

where ∂D is the positively oriented boundary of D

- By Green's Theorem,

$$\begin{aligned}\int_{\partial D} -y\,dx + x\,dy &= \int_D (x)_x - (-y)_y\,dA \\ &= 2\int_D dA = 2(\text{area of } D) \\ &= 2((\text{area of square}) - 2(\text{area of each disk})) \\ &= 2(8 - 2(\frac{\pi}{4})) = 16 - \pi\end{aligned}$$

Consequence of Green's Theorem



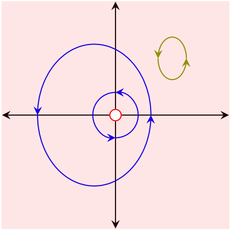
- If $\vec{F} = \vec{i}P + \vec{j}Q$ is a vector field on D such that $\vec{\nabla} \times \vec{F} = Q_x - P_y = 0$, then

$$\int_{C_0} \vec{F} \cdot d\vec{r} - \int_{C_1 \cup \cdots \cup C_N} \vec{F} \cdot d\vec{r} = \int_D Q_x - P_y \, dA = 0$$

- Therefore, if $\vec{F}$ is a vector field on D such that $\vec{\nabla} \times \vec{F} = 0$, then

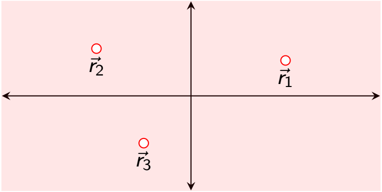
$$\int_{C_0} \vec{F} \cdot d\vec{r} = \int_{C_1 \cup \cdots \cup C_N} \vec{F} \cdot d\vec{r}$$

Fundamental Example of a Curl-Free Vector Field



- The vector field $\vec{F} = \frac{-\vec{i}y+\vec{j}x}{x^2+y^2}$ is defined on 2-space except at the origin
- $\vec{\nabla} \times \vec{F} = 0$
- $\int_C \vec{F} \cdot d\vec{r} = 2\pi$ for any circle C of radius R centered at the origin, oriented counterclockwise
- $\int_C \vec{F} \cdot d\vec{r} = 2\pi$ for any connected closed curve C with the origin inside, oriented counterclockwise
- $\int_C \vec{F} \cdot d\vec{r} = 0$ for any connected closed curve C without the origin inside

Curl-Freee Vector Field on Domain With Holes



- Given points $(x_1, y_1), \ldots, (x_N, y_N)$, consider the vector fields

$$\vec{F}_k = \frac{-\vec{i}(y - y_k) + \vec{j}(x - x_k)}{(x - x_k)^2 + (y - y_k)^2}, \ \ k = 1, \ldots, N$$

- For each $k = 1, \ldots, N$, $\vec{\nabla} \times \vec{F}_k = 0$ and a small circle C_k going counterclockwise around (x_k, y_k),

$$\int_{C_k} \vec{F}_k \cdot d\vec{r} = 2\pi$$

- Consider

$$\vec{F} = a_1 \vec{F}_1 + \cdots + a_N \vec{F}_N,$$

where $a_1, \ldots, a_N$ are constant scalars
- $\vec{\nabla} \times \vec{F} = 0$ everywhere except at (x_1, y_1), $\ldots$, (x_N, y_N)
- If C is a closed curve that goes around all of the points (x_1, y_1), $\ldots$, (x_N, y_N), then by Green's Theorem,

$$\int_C \vec{F} \cdot d\vec{r} = 2\pi(a_1 + a_2 + \cdots a_N)$$