MATH-UA 123 Calculus 3: Line Integrals, Fundamental Theorem of Line Integrals, Conservative Vector Fields, Curl of a Vector Field

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November 10, 2021

LIVE TRANSCRIPT
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Line integral of a Vector Field Along an Oriented Curve

- Let $\vec{F}(x, y, z)$ be a vector field on a domain D
  - The formulas for the components of $\vec{F}$ should use the variables x, y, z only
  - All other variables should be treated as constants
- Let C an oriented curve in D with start point $\vec{r}_{\text{start}}$ and end point $\vec{r}_{\text{end}}$
- The line integral of $\vec{F}$ along C is written as

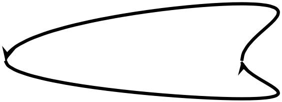
$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz$$

- The line integral of $\vec{F}$ along C is a scalar
- To compute the value of a line integral, do the following:
  - Choose a parameterization of C
    - This gives you formulas for x, y, z in terms of the parameter for C
    - Calculate dx, dy, dz in terms of the parameter and its differential
  - In the integral, replace x, y, z and their differentials by their formulas in terms of the curve parameter and its differential
  - You now have an integral in terms of a single variable, namely the curve parameter
  - Calculate the integral using what you learned in Calculus 2

Properties of Line Integrals

- If C is an oriented curve and $\vec{F}$ is a vector field, then the line integral of $\vec{F}$ along C is
$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=t_{\text{start}}}^{t=t_{\text{end}}} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt,$$
where $\vec{r}(t)$ is a parameterization of C
- The value of the line integral stays the same, even if a different parameterization is used
- Given an oriented curve C, $-C$ will denote the same curve but with the opposite orientation:
$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$
- If $C = C_1 \cup C_2$, then
$$\int_{-C} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

Line Integral Around Closed Curve



- A closed curve is a curve where the start and end points are the same
- A parameterized curve given by
$$\vec{r} : [t_{\text{start}}, t_{\text{end}}] \to \mathbb{R}^n$$
is closed, if $\vec{r}(t_{\text{start}}) = \vec{r}(t_{\text{end}})$
- If C is closed, a line integral
$$\int_C \vec{F} \cdot d\vec{r}$$
is NOT necessarily zero
- Example: Let C be the unit circle centered at the origin, oriented counterclockwise and parameterized as
$$x(t) = \cos t,\ \ y(t) = \sin t,\ \ 0 \le t \le 2\pi,$$
and consider the line integral
$$\int_C -y\, dx + x\, dy = \int_{t=0}^{t=2\pi} -(\sin t)(-\sin t\, dt) + (\cos t)(\cos t)\, dt$$
$$= \int_{t=0}^{t=2\pi} 1\, dt = 2\pi$$

Gradient Field

- A vector field $\vec{F}$ a domain D is a *gradient field*, if there is a scalar function f on D such that
$$\vec{F} = \vec{\nabla} f$$
- Equivalently, a vector field $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3$ is a gradient field if there is a scalar function such that
$$F_1 = f_x, \ F_2 = f_y, \ F_3 = f_z$$
- The function f is called the potential or the energy potential of $\vec{F}$

Fundamental Theorem of Line Integrals

- Let $\vec{F} = \vec{\nabla} f$ be a gradient field on a domain D
- Let C be an oriented curve in D with start point $\vec{r}_{\text{start}}$ and end point $\vec{r}_{\text{end}}$
- We have shown that

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}_{textend}) - f(\vec{r}_{textstart})$$

- If C is a closed curve, then $\vec{r}_{\text{end}} = \vec{r}_{\text{start}}$ and therefore

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

- If C_1 and C_2 are any two oriented curves with the same start and end points, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

Examples

- Let C be the unit circle centered around the origin, going counterclockwise
- $\vec{F} = \langle y, x \rangle$ is a gradient field, since $\vec{F} = \vec{\nabla} f$, where $f(x, y) = xy$

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

- $\vec{G} = \langle -y, x \rangle$ is not a gradient field

$$\int_C \vec{G} \cdot d\vec{r} = \int_C -y\, dx + x\, dy$$
$$= \int_{t=0}^{t=2\pi} -(\sin t)(-\sin t\, dt) + (\cos t)(\cos t)\, dt$$
$$= \int_{t=0}^{t=2\pi} 1\, dt = 2\pi$$

- $\vec{H} = \langle x, y^2 \rangle$ is not a gradient field

$$\int_C \vec{H} \cdot d\vec{r} = \int_C x\, dx + y^2\, dy$$
$$= \int_{t=0}^{t=2\pi} (\cos t)(-\sin t\, dt) + (\sin t)^2 (\cos t)\, dt$$
$$= -\frac{1}{2}(\sin t)^2 + \frac{1}{3}(\sin t)^3 \Big|_{t=0}^{t=2pi}$$
$$= 0$$

Path Independent, Conservative, Gradient Vector Fields

- A vector field $\vec{F}$ is path-independent on a domain D, if, for any two oriented curves C_1 and C_2 in D with the same start points and same end points,
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$
- A vector field $\vec{F}$ is path-independent on a domain D, if, for any closed curve C in D,
$$\int_{C} \vec{F} \cdot d\vec{r} = 0$$
- A vector field $\vec{F}$ is gradient or conservative on a domain D, if there is a potential function f on domain D such that $\vec{\nabla} f = \vec{F}$
- Any path-independent vector field on a domain D is conservative, and any conservative vector field on a domain D is path-independent
- Gradient $\iff$ conservative $\iff$ path-independent

Best Test for a Gradient Field

- Try to solve for the potential function using antidifferentiation
- Consider $\vec{F} = \vec{i}xy^2 + \vec{j}xy^2 + y$
- If $\vec{F}$ is gradient, then there is a function f such that

$$f_x = xy^2 \text{ and } f_y = xy^2 + y$$

- Antidifferentiating the first equation with respect to x, we get

$$f(x,y) = \frac{1}{2}x^2y^2 + g(y)$$

  The "constant term" is allowed to be a function of y because y is a "constant"
- Now test the second equation

$$f_y = x^2y + g'(y) \stackrel{?}{=} xy^2 + y$$

- If we set $g(y) = \frac{1}{2}y^2$ and therefore,

$$f(x,y) = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2,$$

  then

$$\vec{\nabla} f = \langle xy^2, x^2y + y \rangle = \vec{F}$$

Another Example

- Consider $\vec{F} = \langle 3x^2y^2 + 2xy, 2x^3y + 2xy \rangle$
- $f_x = 3x^2y^2 + 2xy \implies f = x^3y^2 + x^2y + g(y)$
- $f_y = 2x^3y + x^2 + g'(y) \stackrel{?}{=} 2x^3y + 2xy$
- No solution exists $\implies \vec{F}$ is not a gradient field

The Curl of a Vector Field

- In dimension 2
  - Recall that the curl of two vectors $\vec{v} = \vec{i}v_1 + \vec{j}v_2$ and $\vec{w} = \vec{i}w_1 + \vec{j}w_2$ is the scalar
  $$\vec{v} \times \vec{w} = v_1 w_2 - v_2 w_1$$
  - The curl of a vector field $\vec{F} = \vec{i}F_1 + \vec{j}F_2$ is the scalar function
  $$\vec{\nabla} \times \vec{F} = (\vec{i}\partial_x + \vec{j}\partial_y) \times (\vec{i}F_1 + \vec{j}F_2) = (F_2)_x - (F_1)_y$$
- In dimension 3
  - Recall that the curl of two vectors $\vec{v} = \vec{i}v_1 + \vec{j}v_2 + \vec{k}v_3$ and $\vec{w} = \vec{i}w_1 + \vec{j}w_2 + \vec{k}w_3$ is the vector
  $$\vec{v} \times \vec{w} = \vec{i}(v_2 w_3 - v_3 w_2) + \vec{k}(v_3 w_1 - v_1 w_3) + \vec{k}(v_1 w_2 - v_2 w_1)$$
  - The curl of a vector field $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{j}F_3$ is the vector field
  $$\begin{aligned}\vec{\nabla} \times \vec{F} &= (\vec{i}\partial_x + \vec{j}\partial_y + \vec{k}\partial_z) \times (\vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3) \\ &= \vec{i}((F_3)_y - (F_2)_z) + \vec{j}((F_1)_z - (F_3)_x) \\ &\quad + \vec{k}((F_2)_x - (F_1)_y)\end{aligned}$$

Curl Test for Gradient Field

- In dimension 2: If $\vec{F} = \vec{\nabla} f$, then

$$\vec{\nabla} \times \vec{F} = (F_2)_x - (F_1)_y = (f_y)_x - (f_x)_y = 0$$

- In dimension 3: If $\vec{F} = \vec{\nabla} f$, then

$$\begin{aligned}\vec{\nabla} \times \vec{F} &= \vec{i}((F_3)_y - (F_2)_z) + \vec{j}((F_1)_z - (F_3)_x) + \vec{k}((F_2)_x - (F_1)_y) \\ &= \vec{i}((f_z)_y - (f_x)_y) + \vec{j}((f_x)_z - (f_z)_x) + \vec{k}((f_y)_x - (f_x)_y) \\ &= \vec{0}\end{aligned}$$

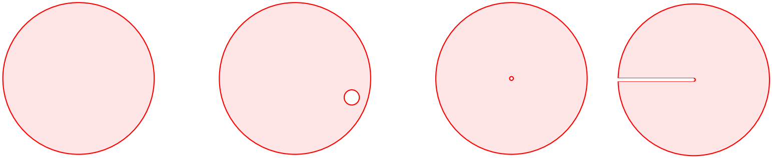
- In either dimension: If $\vec{F}$ is a gradient field, then $\vec{\nabla} \times \vec{F} = 0$
- BEWARE: The converse is not necessarily true: If $\vec{F} = \vec{i}F_1 + \vec{j}F_2$ satisfies

$$\partial_y F_1 = \partial_x F_2,$$

it does not necessarily imply that $\vec{F}$ is a gradient field

Simply Connected Domain

- A domain D in 2-space is called **simply connected**, if, given any closed curve C in D, all points inside C are in D



Simply connected | Not simply connected | Not simply connected | Simply connected

- A domain D in 3-space is **simply connected** if, given any closed curve C in D, there is a surface in D whose boundary is C
  - 3-space with the origin removed is simply connected
  - 3-space with the z-axis removed is not simply connected

Curl Test on a Simply Connected Domain

- If $\vec{F}$ is a vector field on a simply connected domain D and
$$\vec{\nabla} \times \vec{F} = 0,$$
then $\vec{F}$ is a gradient field on D
- If $\vec{F}$ is a vector field on a simply connected domain D and
$$\vec{\nabla} \times \vec{F} = 0,$$
then there is a potential function f such that $\vec{F} = \vec{\nabla} f$

Fundamental Example

- $\vec{F}(x, y) = \dfrac{-\vec{i}y + \vec{j}x}{x^2 + y^2}$
- $\vec{F}$ is undefined at the origin
- Suppose C is a circle centered at the origin oriented counterclockwise
- A parameterization of C is

$$\vec{r}(t) = R\langle \cos(t), \sin(t)\rangle, \text{ where } R = \text{ radius of circle}$$

- The line integral around a circle C centered at the origin is

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_{t=0}^{t=2\pi} \vec{F}(x(t), y(t)) \cdot \langle x'(t), y'(t)\rangle \, dt \\
&= \int_{t=0}^{t=2\pi} \left(\frac{-\vec{i}y + \vec{j}x}{x^2 + y^2} \right) \cdot (\vec{i}x' + \vec{j}y') \, dt \\
&= \frac{1}{R^2} \int_{t=0}^{t=2\pi} -yx' + xy' \, dt \\
&= \frac{1}{R^2} \int_{t=0}^{t=2\pi} (R\sin(t))^2 + (R\cos(t))^2 \, dt \\
&= 2\pi
\end{aligned}$$

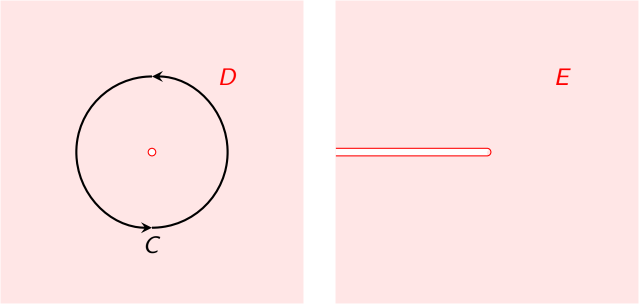
- Therefore, $\vec{F}$ is not conservative on 2-space with the origin removed

Curl of Fundamental Example

- $\vec{F} = \dfrac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$
- The curl of $\vec{F}$ is

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= -\partial_y \left(\frac{-y}{x^2 + y^2} \right) + \partial_x \left(\frac{x}{x^2 + y^2} \right) \\
&= \frac{1}{x^2 + y^2} - \frac{y(2y)}{(x^2 + y^2)^2} \frac{1}{x^2 + y^2} - \frac{x(2x)}{(x^2 + y^2)^2} \\
&= \frac{2(x^2 + y^2) - 2y^2 - 2x^2}{(x^2 + y^2)^2} \\
&= 0
\end{aligned}$$

Fundamental Example



Not simply connected | Simply connected

- $\vec{F}(x,y) = \dfrac{-\vec{i}y + \vec{j}x}{x^2+y^2}$
- $\vec{\nabla} \times \vec{F} = 0$
- $\vec{F}$ on D is not a gradient field, because

$$\int_C \vec{F} \cdot d\vec{r} = 2\pi \neq 0,$$

- $\vec{F}$ on E is a gradient field, because E is simply connected

Fundamental Example

- Consider the function $f(x, y) = \theta$, where

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ r = \sqrt{x^2 + y^2}$$

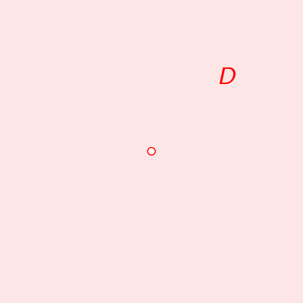
- Compute gradient of f implicitly

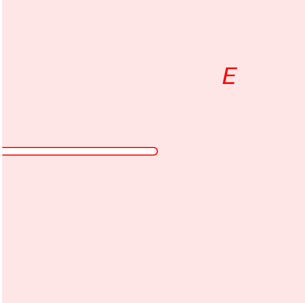
$$\begin{aligned}
\langle x, y\rangle &= \langle r\cos\theta, r\sin\theta\rangle \\
\langle dx, dy\rangle &= \langle dr\ \cos\theta - d\theta\, r\sin\theta, dr\ \sin\theta + d\theta\, r\cos\theta\rangle \\
&= dr\langle\cos\theta, \sin\theta\rangle + d\theta\langle -r\sin\theta, r\cos\theta\rangle \\
&= \frac{dr}{r}\langle x, y\rangle + d\theta\langle -y, x\rangle \\
-y\,dx + x\,dy &= d\theta\,(x^2 + y^2) \\
d\theta &= -\frac{-y, dx + x\,dy}{x^2 + y^2}
\end{aligned}$$

- Therefore,

$$\vec{\nabla} f = \vec{i} f_x + \vec{j} f_y = \vec{i}\theta_x + \vec{j}\theta_y = \frac{-y, dx + x\,dy}{x^2 + y^2} = \vec{F}$$

The Fundamental Example Is Not And Is A Gradient Field





- $\vec{F}(x, y) = \dfrac{-\vec{i}y + \vec{j}x}{x^2 + y^2}$
- If $f(x, y) = \theta$, then $\vec{F} = \vec{\nabla} f$
- There is no way to define θ as a smooth function on D
- There is, however, a way to define θ as a smooth function on E
  - For each (x, y) not on the negative x-axis, define θ so that
    $$-\pi < \theta < \pi$$
  - For each (x, y), there is a unique such θ satisfying this