MATH-UA 123 Calculus 3:
Line Integrals, Fundamental Theorem of Line Integrals

Deane Yang

Courant Institute of Mathematical Sciences
New York University

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LIVE TRANSCRIPT

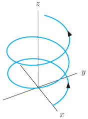
START RECORDING

Parameterized Curves

- Recall that a parameterized curve is a map from an interval into 2-space or 3-space,

$$c : I \to \mathbb{R}^n, \text{ where } n = 2 \text{ or } 3$$

- The velocity of c is $\vec{v}(t) = c'(t)$
- We will assume that the velocity is always nonzero
- The path of the curve is the image of c

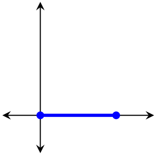


- A path has many different parameterizations
- The parameterized curves

$$c_1(t) = (t, 0),\ 0 \leq t \leq 1$$
$$c_2(t) = (t, 0),\ 0 \leq t \leq 1$$
$$c_3(t) = (1 - t, 0),\ 0 \leq t \leq 1$$

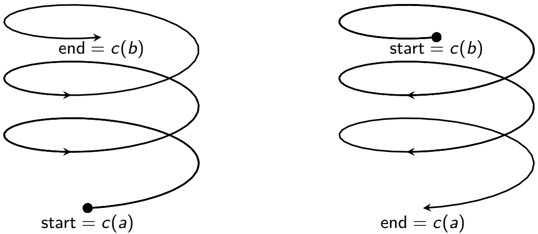
have the same path

Same Path, Different Parameterizations



- $c_1 : [0, 1] \to \mathbb{R}^2$, where
$$c_1(s) = s$$
- $c_1 : [-1, 0] \to \mathbb{R}^2$, where
$$c_1(s) = -s$$
- $c_1 : [0, 1] \to \mathbb{R}^2$, where
$$c_1(s) = 1 - s$$
- $c_1 : [0, 1] \to \mathbb{R}^2$, where
$$c_1(s) = s^2$$

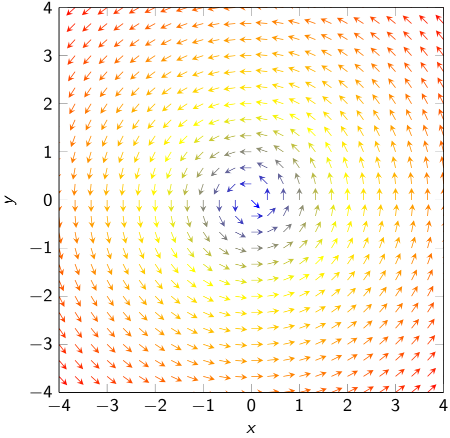
Oriented Curve



- Orientation of a parameterized curve is direction of travel
- There are two possible orientations
  - The direction of the velocity vector
  - The opposite direction to the velocity vector
- Consider a curve $c : [a, b] \to \mathbb{R}^n$
- If the orientation is in the direction of the velocity vector $c'(t)$, then $c(a)$ is the start point and $c(b)$ is the end point
- If the orientation is in the opposite direction of the velocity vector $c'(t)$, then $c(b)$ is the start point and $c(a)$ is the end point

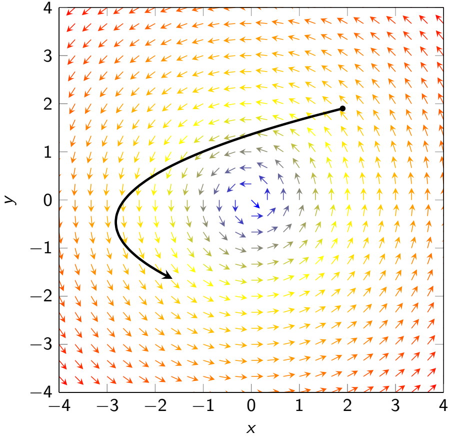
Vector Field

$$\vec{V}(x, y) = \langle v_1(x, y), v_2(x, y) \rangle$$



Oriented Curve in Vector Field

$$\vec{r}(t) = \langle x(t), y(t) \rangle,\ a \le t \le b$$



Oriented Curve in Vector Field

- Chop interval $[a, b]$ into N equal pieces

$$\Delta t = \frac{b-a}{N}$$

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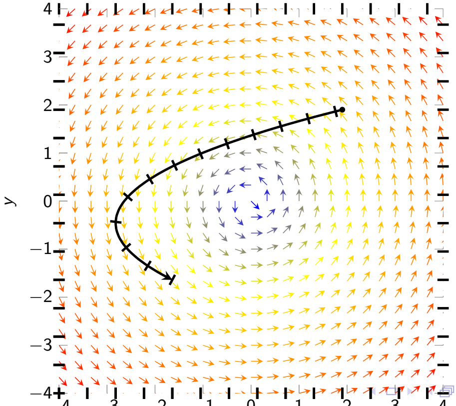
$$a = t_1 < t_2 = t_1 + \Delta t < \cdots < t_N = t_1 + (N-1)\Delta t < t_{N+1} = t_1 + (N+1)\Delta t = b$$

- Linear approximation of curve

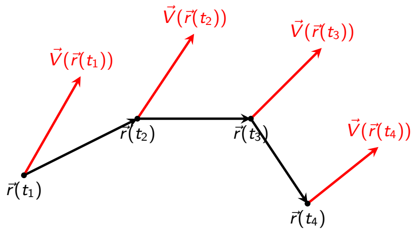
$$\vec{r}(t_k) - \vec{r}(t_{k-1}) \simeq \vec{r}'(t_k)(t_k - t_{k-1})$$



Line Integral of Vector Field Along Oriented Curve

- Let C be a curve in a domain D with parameterization $\vec{r}(t)$, for each t between a and b
- Let $\vec{V}$ be a vector field on the domain D
- Define the line integral of a vector field $\vec{V}$ along an oriented curve C to be

$$\begin{aligned}\int_C \vec{V}\cdot d\vec{r} &\simeq \vec{V}(\vec{r}(t_1))\cdot(\vec{r}(t_2)-\vec{r}(t_1))+\cdots+\vec{V}(\vec{r}(t_N))\cdot(\vec{r}(t_{N+1})-\vec{r}(t_N))\\ &\simeq \vec{V}(\vec{r}(t_1))\cdot\vec{r}\,'(t_1)(t_2-t_1)+\cdots+\vec{V}(\vec{r}(t_N))\cdot(\vec{r}\,'(t_N)(t_{N+1}-t_N)\\ &\to \int_{t=a}^{t=b}\vec{V}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt,\end{aligned}$$



- Does not matter whether $a \le b$ or $a \ge b$

Line integral of a Vector Field Along an Oriented Curve

- Let $\vec{F}(x, y, z)$ be a vector field on a domain D
- Let C an oriented curve in D with start point $\vec{r}_{\text{start}}$ and end point $\vec{r}_{\text{end}}$
- Let $\vec{r}(t)$ be a parameterization of C such that

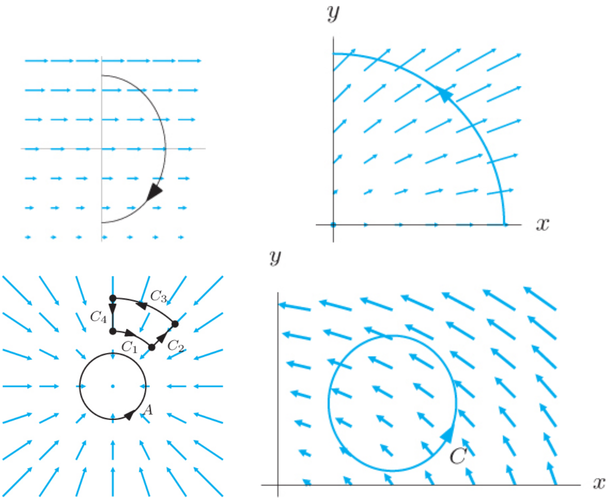
$$\vec{r}(t_{\text{start}}) = \vec{r}_{\text{start}} \text{ and } \vec{r}(t_{\text{end}}) = \vec{r}_{\text{end}}$$

- The line integral of $\vec{F}$ along the curve C is defined to be

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=t_{\text{start}}}^{t=t_{\text{end}}} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

- Here, $\vec{F} = \vec{F}(\vec{r}(t))$ and $d\vec{r} = \vec{r}'(t)\, dt$

Examples

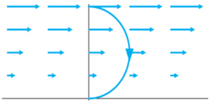


Calculation of Line Integral in Constant Vector Field

- Consider a constant vector field $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3$, where F_1, F_2, F_3 are scalar constants
- An oriented curve C with parameterization $\vec{r}(t) = \vec{i}x(t) + \vec{j}y(t) + \vec{j}z(t)$, $a \leq t \leq b$, oriented in the direction of the velocity vector
  - $d\vec{r} = \vec{r}'(t)\,dt = (\vec{i}x'(t) + \vec{j}y'(t) + \vec{k}z'(t))\,dt$
- We want to compute the line integral of $\vec{F}$ along the oriented curve C

$$
\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_{t=a}^{t=b} \langle F_1, F_2, F_3\rangle \langle x'(t), y'(t), z'(t)\rangle\,dt \\
&= \int_{t=a}^{t=b} F_1x'(t) + F_2y'(t) + F_3z'(t)\,dt \\
&= F_1x(t) + F_2y(t) + F_3z(t)\big|_{t=a}^{t=b} \\
&= F_1(x(b) - x(a)) + F_2(y(b) - y(a)) + F_3(z(b) - z(a)) \\
&= \vec{F} \cdot (\vec{r}(b) - \vec{r}(a))
\end{aligned}
$$

Example of Line Integral



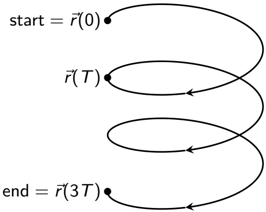
- Vector field: $\vec{F}(x,y) = y\vec{i}$
- Curve C: $\vec{r}(t) = \vec{i}\cos(t) + \vec{j}(1+\sin(t))$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, oriented opposite to the velocity
- $d\vec{r} = (-\vec{i}\sin(t) + \vec{j}\cos(t))\, dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=\frac{\pi}{2}}^{t=-\frac{\pi}{2}} (\sin(t)\vec{i}) \cdot (-\vec{i}\sin(t) + \vec{j}\cos(t))\, dt$$

$$= \int_{t=\frac{\pi}{2}}^{t=-\frac{\pi}{2}} -(\sin(t))^2\, dt = \int_{t=\frac{\pi}{2}}^{t=-\frac{\pi}{2}} \frac{-1+\cos(2t)}{2}\, dt$$

$$= -\frac{t}{2} + \frac{\sin(2t)}{4}\Bigg|_{t=\frac{\pi}{2}}^{t=-\frac{\pi}{2}} = \frac{\pi}{2}$$

Work Done By Gravity Along Helical Path



- $\vec{r}(t) = \langle R\cos\left(\frac{2\pi t}{T}\right), R\sin\left(\frac{2\pi t}{T}\right), h(t)\rangle$, $0 \le t \le 3T$, where

$$T = \text{ period and } h(t) = \text{height at time } t \text{ (meters)}$$

- Work done by gravity $\vec{F} = -g\vec{k}$

$$\int_C \vec{F}\,d\vec{r} = \int_{t=0}^{t=3T} -g \cdot \langle -\frac{2\pi R}{T}\sin\left(\frac{2\pi t}{T}\right), R\frac{2\pi}{T}\cos\left(\frac{2\pi t}{T}\right), -h'(t)\rangle\,dt$$

$$= \int_{t=0}^{t=3T} gh'(t)\,dt = g(h(3T))$$

Another Notation for a Line Integral

- Consider a vector $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3$ and a parameterized curve $\vec{r}(t) = \vec{i}x(t) + \vec{j}y(t) + \vec{k}z(t)$
- $d\vec{r} = \vec{i}dx + \vec{j}dy + \vec{k}dz$
- $\vec{F} \cdot d\vec{r} = F_1\,dx + F_2\,dy + F_3\,dz$
- The line integral of $\vec{F}$ along an oriented curve C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

- Example: Suppose C is parameterized by

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle,\ \ 0 \le t \le 1$$

and we want to compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_C x\,dx + y\,dy + z\,dz$$

- Since $x = t,\ y = t^2,\ z = t^3$,

$$dx = dt,\ \ dy = 2t\,dt,\ \ dz = 3t^2\,dt$$

- Therefore,

$$\int_C x\,dx + y\,dy + z\,dz = \int_{t=0}^{t=1} t\,dt + t^2(2t\,dt) + t^3(3t^2)\,dt$$

$$= \int_{t=0}^{t=1} (t + 2t^3 + 3t^5)\,dt = \frac{3}{2}$$

Example of Line Integral in 2-space

- Suppose C is an oriented curve in 2-space with parameterization $\vec{r}(t) = \vec{i}x(t) + \vec{j}y(t)$, $a \le t \le b$, and $P(x,y)$, $Q(x,y)$ are scalar functions
- To compute the line integral $\int_C P\,dx + Q\,dy$,

$$\int_C P\,dx + Q\,dy = \int_{t=a}^{t=b} (P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t))\,dt$$

- Example: Suppose the curve C has the parameterization $\vec{r}(t) = \vec{i}t\cos(t) + \vec{j}t\sin(t)$, $0 \le t \le 2\pi$, with orientation in the direction of the velocity vector
- To calculate $\int_C -y\,dx + x\,dy$,

$$\begin{aligned}\int_C -y\,dx + x\,dy &= \int_{t=0}^{t=2\pi} -(t\sin t)(\cos(t) - t\sin(t))\,dt \\ &\quad + (t\cos(t))(\sin(t) + t\cos(t))\,dt \\ &= \int_{t=0}^{t=2\pi} t^2\,dt = \frac{(2\pi)^3}{3}\end{aligned}$$

Properties of Line Integrals

- If C is an oriented curve and $\vec{F}$ is a vector field, then the line integral of $\vec{F}$ along C is
$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=t_{\text{start}}}^{t=t_{\text{end}}} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt,$$
where $\vec{r}(t)$ is a parameterization of C
- The value of the line integral stays the same, even if a different parameterization is used
- Given an oriented curve C, $-C$ will denote the same curve but with the opposite orientation:
$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$
- If $C = C_1 \cup C_2$, then
$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

Gradient Field

- A vector field $\vec{F}$ a domain D is a *gradient field*, if there is a scalar function f on D such that
$$\vec{F} = \vec{\nabla} f$$
- Equivalently, a vector field $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3$ is a gradient field if there is a scalar function such that
$$F_1 = f_x, \; F_2 = f_y, \; F_3 = f_z$$
- The function f is called the potential or the energy potential of $\vec{F}$
- $\vec{F} = \langle x, y, z \rangle$ is a gradient field, because $\vec{F} = \nabla f$, where
$$f(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$$
- $\vec{G} = \langle y, x \rangle$ is a gradient field, because $\vec{G} = \nabla q$, where
$$q(x, y, z) = xy$$
- $\vec{H} = \langle y, -x \rangle$ is not a gradient field, because if $\vec{H} = \langle y, -x \rangle = \nabla p = \langle p_x, p_y \rangle$, then
$$p_x = y \text{ and } p_y = -x, \text{ which implies } p_{xy} = 1 \text{ and } p_{yx} = -1$$

Test for a Gradient Field: Partial Derivatives Commute

- If $\vec{F} = \vec{i}F_1 + \vec{j}F_2 = \nabla f = \vec{i}f_x + \vec{j}f_y$, then

$$\partial_y F_1 = (f_x)_y = (f_y)_x = \partial_x F_2$$

- If $\vec{F} = \vec{i}F_1 + \vec{j}F_2 + \vec{k}F_3 = \nabla f = \vec{i}f_x + \vec{j}f_y + \vec{k}f_z$, then

$$\begin{aligned}\partial_y F_1 &= (f_x)_y = (f_y)_x = \partial_x F_2\\ \partial_z F_1 &= (f_x)_z = (f_z)_x = \partial_x F_3\\ \partial_z F_2 &= (f_y)_z = (f_z)_y = \partial_y F_3\end{aligned}$$

- BEWARE: The converse is not necessarily true: If $\vec{F} = \vec{i}F_1 + \vec{j}F_2$ satisfies

$$\partial_y F_1 = \partial_x F_2,$$

it does not necessarily imply that $\vec{F}$ is a gradient field

Line Integral of Gradient Field

- Suppose $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a parameterization of an oriented curve that starts at $\vec{r}_{\text{start}} = \vec{r}(t_{\text{start}})$ and ends at $\vec{r}_{\text{end}} = \vec{r}(t_{\text{end}})$
- Consider the scalar function $\phi(t) = f(\vec{r}(t))$
- By the chain rule,

$$\begin{aligned} \phi'(t) &= \frac{d}{dt}(f(\vec{r}(t)) \\ &= f_x x' + f_y y' + f_z z' \\ &= \langle f_x, f_y, f_z \rangle \cdot \langle x', y', z' \rangle \\ &= \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}'(t) \end{aligned}$$

- Therefore, by the definition of the line integral and the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_C \vec{\nabla} f \cdot d\vec{r} &= \int_{t=t_{\text{start}}}^{t=t_{\text{end}}} \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \\ &= \int_{t=t_{\text{start}}}^{t=t_{\text{end}}} \phi'(t)\, dt \\ &= \phi(t_{\text{end}}) - \phi(t_{\text{start}}) \\ &= f(\vec{r}(t_{\text{end}})) - f(\vec{r}(t_{\text{start}})) \\ &= f(\vec{r}_{\text{end}}) - f(\vec{r}_{\text{start}}) \end{aligned}$$

Fundamental Theorem of Line Integrals

- Let $\vec{F} = \vec{\nabla} f$ be a gradient field on a domain D
- Let C be an oriented curve in D with start point $\vec{r}_{\text{start}}$ and end point $\vec{r}_{\text{end}}$
- We have shown that

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}_{textend}) - f(\vec{r}_{textstart})$$

- If C is a closed curve, then $\vec{r}_{\text{end}} = \vec{r}_{\text{start}}$ and therefore

$$\int_C \vec{F} \cdot d\vec{r} = 0$$

- If C_1 and C_2 are any two oriented curves with the same startand end points, then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

Path Independent, Conservative, Gradient Vector Fields

- A vector field $\vec{F}$ is path-independent on a domain D, if, for any two oriented curves C_1 and C_2 in D with the same start points and same end points,
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$
- A vector field $\vec{F}$ is path-independent on a domain D, if, for any closed curve C in D,
$$\int_{C} \vec{F} \cdot d\vec{r} = 0$$
- A vector field $\vec{F}$ is gradient or conservative on a domain D, if there is a potential function f on domain D such that $\vec{\nabla} f = \vec{F}$
- Any path-independent vector field on a domain D is conservative, and any conservative vector field on a domain D is path-independent
- Gradient $\iff$ conservative $\iff$ path-independent