MATH-UA 123 Calculus 3: Rectangular, Polar, Cylindrical, Spherical Coordinates Vector Fields, Line Integrals

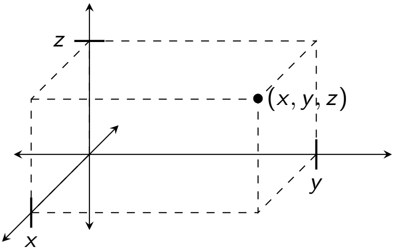
Deane Yang

Courant Institute of Mathematical Sciences
New York University

November 3, 2021

START RECORDING

Rectangular Coordinates in 2-Space and 3-Space



- Ranges

$$-\infty < x, y, z < \infty$$

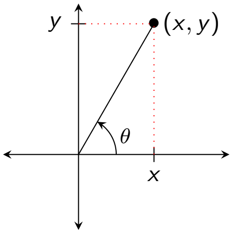
- Area form

$$dA = dx\,dy$$

- Volume form

$$dV = dx\,dy\,dz$$

Polar Coordinates



- Ranges

$$r \geq 0$$
$$0 \leq \theta \leq 2\pi \text{ or } -\pi \leq \theta \leq \pi$$

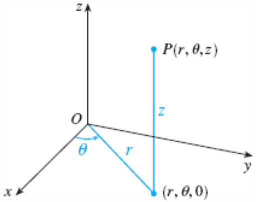
- Coordinate formulas

$$x = r\cos\theta$$
$$y = r\sin\theta$$

- Area form

$$dA = r\,dr\,d\theta$$

Cylindrical Coordinates



- Ranges

$$r \geq 0$$
$$0 \leq \theta \leq 2\pi \text{ or } -\pi \leq \theta \leq \pi$$
$$-\infty < z < \infty$$

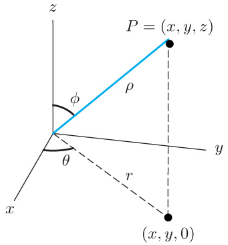
- Coordinate formulas

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

- Volume form

$$dV = r\,dr\,d\theta\,dz$$

Spherical Coordinates



- Ranges

$$\rho \geq 0$$
$$0 \leq \phi \leq \pi$$
$$0 \leq \theta \leq 2\pi \text{ or } -\pi \leq \theta \leq \pi$$

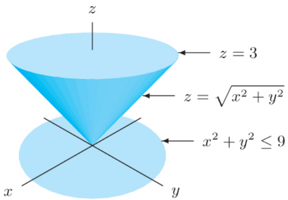
- Coordinate formulas

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

- Volume form

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Cone



- Rectangular coordinates

$$x^2 + y^2 \leq 9$$
$$\sqrt{x^2 + y^2} \leq z \leq 3$$

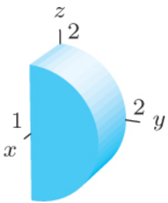
- Cylindrical coordinates

$$0 \leq \theta \leq 2\pi$$
$$r \leq 3$$
$$r \leq z \leq 3$$

- Spherical coordinates

$$0 \leq \theta \leq 2\pi$$
$$0 \leq \phi \leq \frac{\pi}{2}$$
$$0 \leq \rho \leq \frac{3}{\cos \phi}$$

Half Cylinder



- Rectangular coordinates

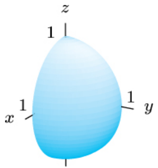
$$\begin{aligned} y^2 + z^2 &\leq 4 \\ y &\geq 0 \\ 0 &\leq x \leq 1 \end{aligned}$$

- Cylindrical coordinates

$$\begin{aligned} y &= r\cos\theta, \; z = r\sin\theta, \; x = x \\ -\frac{\pi}{2} &\leq \theta \leq \frac{\pi}{2} \\ r &\leq 2 \\ 0 &\leq x \leq 1 \end{aligned}$$

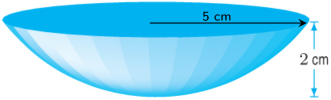
- Spherical coordinates: Too complicated

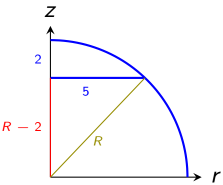
Triple Integral Over Quarter Sphere



- Rectangular or cylindrical coordinates:
  - Region R is between the graphs $z = -\sqrt{1-x^2-y^2}$ and $z = \sqrt{1-x^2-h^2}$ over the 2D domain $D = \{x^2+y^2 \leq 1 \text{ and } x, y, \geq 0\}$
  - Integral over region is $\int_R f\, dV = \int_D \int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} f(x,y,z)\, dx\, dA$
  - In cylindrical coordinates this is equal to $\int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=1} \int_{z=-\sqrt{1-r^2}}^{z=\sqrt{1-r^2}} f(r\cos\theta, r\sin\theta, z)\, dz\, r\, dr\, d\theta$
- Spherical coordinates
  - $R = \{0 \leq \rho \leq 1,\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq \frac{\pi}{2}\}$
  - $\int_R f\, dV =$
    $\int_{\rho=0}^{\rho=1} \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=\frac{\pi}{2}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, d\theta\, \sin\phi\, d\phi\, \rho^2\, d\rho$

Volume of Spherical Cap





- Find radius R of sphere

$$(R-2)^2+5^2=R^2 \implies R^2-4R+4+25=R^2 \implies R=\frac{29}{4}$$

- In cylindrical coordinates

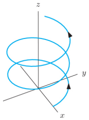
$$V=\int_C dV=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=5}\int_{z=R-2}^{z=\sqrt{R^2-r^2}} dz\, r\, dr\, d\theta$$

- In spherical coordinates, if $R\sin\Phi=5$, then

$$V=\int_C dV=\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\Phi}\int_{\rho=\frac{5}{\cos\phi}}^{\rho=R} \rho^2\, d\rho\, \sin(\phi)\, d\phi\, d\theta$$

Parameterized Curves

- Recall that a parameterized curve is a map from an interval into 2-space or 3-space,
$$c : I \to \mathbb{R}^n, \text{ where } n = 2 \text{ or } 3$$
- The velocity of c is $\vec{v}(t) = c'(t)$
- We will assume that the velocity is always nonzero
- The path of the curve is the image of c

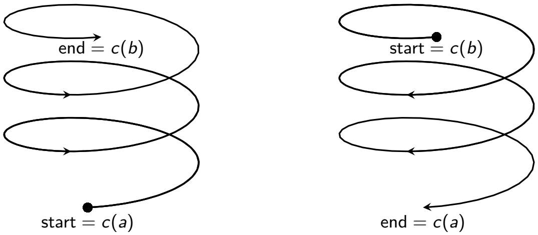


- Two different parameterized curves can have the same path
- The parameterized curves

$$\begin{aligned} c_1(t) &= (t, 0), \ 0 \le t \le 1 \\ c_2(t) &= (t, 0), \ 0 \le t \le 1 \\ c_3(t) &= (1 - t, 0), \ 0 \le t \le 1 \end{aligned}$$

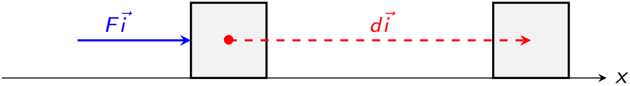
have the same path

Oriented Curve



- Orientation of a parameterized curve is direction of travel
- There are two possible orientations
  - The direction of the velocity vector
  - The opposite direction to the velocity vector
- If the orientation is in the direction of the velocity vector, then $c(a)$ is the start point and $c(b)$ is the end point
- If the orientation is in the opposite direction of the velocity vectorm then $c(b)$ is the start point and $c(a)$ is the end point

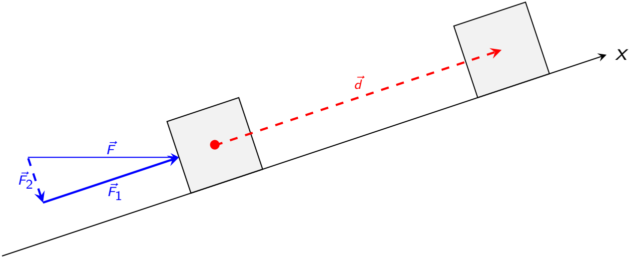
Work or Energy



- Assume no friction, air resistance, or other forces
- If an object is moved horizontally a distance d by a constant horizontal force F, then the energy or work used is

$$E = Fd$$

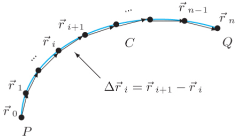
Work or Energy

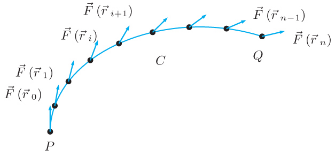


- Assume no gravity, friction, air resistance, or other forces
- If a force $\vec{F}$ acting an object sitting on a flat surface moves the object by the displacement vector $\vec{d}$, then the energy or work used is

$$E = |\vec{F}_1||\vec{d}| = (c_{\vec{d}/|\vec{d}|}\vec{F})|\vec{d}| = \left(\vec{F} \cdot \frac{\vec{d}}{|\vec{d}|}\right)|\vec{d}| = \vec{F} \cdot \vec{d}$$

Object Moving Through Force Field





- Suppose $\vec{r}(t)$, $a \leq t \leq b$, is the path of an object through a force field $\vec{F}(\vec{r})$
- Suppose $a = t_0 < t_1 \cdots < t_n = b$
- The energy used is

$$\begin{aligned} E &\simeq \vec{F}(\vec{r}(t_0)) \cdot (\vec{r}(t_1) - \vec{r}(t_0)) + \cdots + \vec{F}(\vec{r}(t_{n-1})) \cdot (\vec{r}(t_n) - \vec{r}(t_{n-1})) \\ &\simeq \vec{F}(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0)(t_1 - t_0) + \cdots \vec{F}(\vec{r}(t_{n-1})) \cdot \vec{r}\,'(t_{n-1})(t_n - t_{n-1}) \end{aligned}$$

- If we take the limit, we get

$$E = \int_{t=a}^{t=b} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

Line integral of a Vector Field Along an Oriented Curve

- Suppose $\vec{F}(x, y, z)$ is a vector field and C is an oriented curve
- Suppose $\vec{r}(t)$, $a \leq t \leq b$, is a parameterization of C, where the orientation of the curve is in the direction of $\vec{r}'(t)$
- The line integral of $\vec{F}$ along the curve is defined to be

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=a}^{t=b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

- Here, $d\vec{r} = \vec{r}'(t)\, dt$
- If the orientation of C is opposite the direction of $\vec{r}'(t)$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=b}^{t=a} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

- Either way, if $c(t_{\text{start}})$ is the start of C and $c(t_{\text{end}})$ is the end of C, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=t_{\text{start}}}^{t=t_{\text{end}}} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$