MATH-UA 123 Calculus 3: Integration using Polar, Cylindrical, Spherical Coordinates

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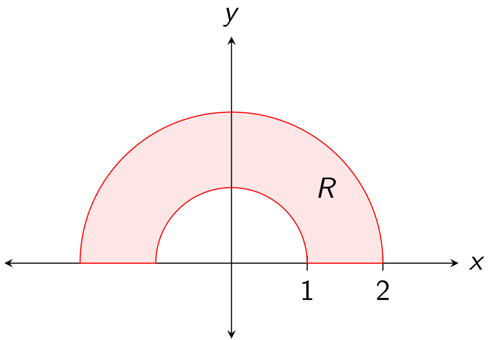
START RECORDING

Integration Over All or Part of Circular Region



- Suppose we want to integrate

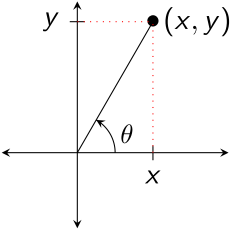
$$\int_R f(x,y)\,dA$$

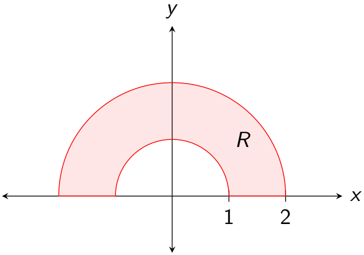
where

$$R = \{1 \leq x^2 + y^2 \leq 4 \text{ and } y \geq 0\}$$

- R is not easily written as the region between two graphs

Polar Coordinates





$$r^2 = x^2 + y^2$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$R = \{(r\cos\theta, r\sin\theta) \;:\; 1 \le r \le 2 \text{ and } 0 \le \theta \le \pi\}$$

- Ranges of values for polar coordinates

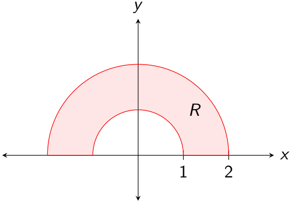
$$r \ge 0$$
$$0 \le \theta \le 2\pi \text{ or } -\pi \le \theta \le \pi$$

Calculation of Integral Using (Reverse) Substitution

- Calculate
$$\int_{x=-2}^{x=2} \frac{dx}{\sqrt{4-x^2}}$$
- If $x = 2\cos\theta$, then
  - $-2 \le x \le 2$ corresponds to $\pi \ge \theta \ge 0$
  - $dx = -2\sin\theta\, d\theta$
  - $\sqrt{4-x^2} = 2\sin\theta$ (nonnegative if $\pi \ge \theta \ge 0$)
- Therefore, the integral can be rewritten as

$$\begin{aligned}\int_{x=-2}^{x=2} \frac{dx}{\sqrt{4-x^2}} &= \int_{\theta=\pi}^{\theta=0} \frac{-2\sin\theta\, d\theta}{2\sin\theta} \\ &= -\int_{\theta=\pi}^{\theta=0} d\theta \\ &= -(0-\pi) = \pi\end{aligned}$$

Calculate Double Integral Using Substitution



- Suppose we want to calculate using polar coordinates

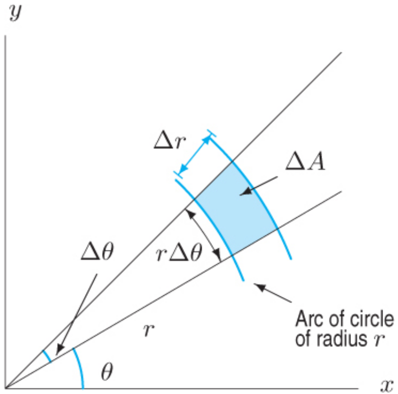
$$\int_R y\,dA = \int_{r=1}^{r=2} \int_{\theta=0}^{\theta=\pi} r \sin\theta (\ldots)\, d\theta\, dr$$

- **IMPORTANT:** $dA \neq d\theta\, dr$
- Using xy–coordinates, it is the area of a small rectangle formed by a small change dx in x and a small change dy in y

$$dA = dx\, dy$$

- Using polar coordinates, it is the roughly area of of a small region formed by a small change dr in r and a small change $d\theta$ in θ

dA in Polar Coordinates

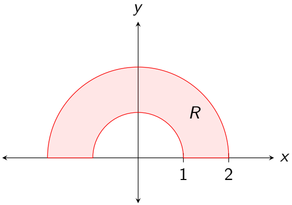


- In polar coordinates

$$dA = (r\,d\theta)(dr) = r\,dr\,d\theta$$

- dA gets bigger as r increases

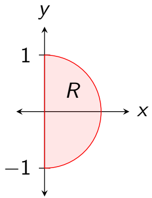
Double Integral in Polar Coordinates



Use $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$:

$$\begin{aligned}
\int_R y\,dA &= \int_{r=1}^{r=2}\int_{\theta=0}^{\theta=\pi} (r\sin\theta)\,r\,dr\,d\theta \\
&= \left(\int_{r=1}^{r=2} r^2\,dr\right)\left(\int_{\theta=0}^{\theta=2} \sin\theta\,d\theta\right) \\
&= \left(\left.\frac{r^3}{3}\right|_{r=1}^{r=2}\right)\left(-\cos\theta\Big|_{\theta=0}^{\theta=\pi}\right) \\
&= \frac{14}{3}
\end{aligned}$$

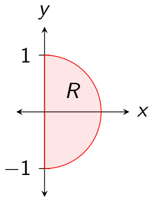
Example of Double Integral in Polar Coordinates



- Compute $\int_R (x^2 + y^2)^{3/2}\, dA$ using xy coordinates
- R is the region between the graphs $x = 0$ and $x = \sqrt{1 - y^2}$, where $-1 \le y \le 1$
- Integral is therefore

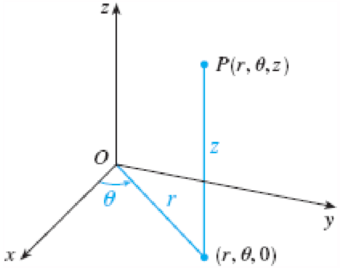
$$\int_R (x^2 + y^2)^{3/2}\, dA = \int_{y=-1}^{y=1} \left(\int_{x=0}^{x=\sqrt{1-y^2}} (x^2 + y^2)^{3/2}\, dx \right) dy$$

Example of Double Integral in Polar Coordinates



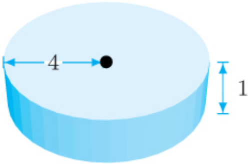
- Compute $\int_R (x^2+y^2)^{3/2}\,dA$ using polar coordinates
- $R = \{(r\cos\theta, r\sin\theta) \;:\; 0 \le r \le 1 \text{ and } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$
- $$\begin{aligned}\int_R (x^2+y^2)^{3/2}\,dA &= \int_{r=0}^{r=1}\left(\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} r^3 r\,d\theta\right)dr\\ &= \left(\int_{r=0}^{r=1} r^4\,dr\right)\left(\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} d\theta\right)\\ &= \frac{\pi}{5}\end{aligned}$$

Cylindrical coordinates in 3-Space



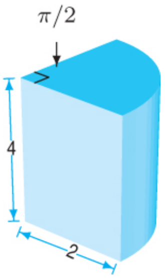
- Use polar coordinates for x and y, and keep z unchanges
- $(x, y, z) = (r\cos\theta, r\sin\theta, z)$
- Convert $(3, 3, 5)$ to cylindrical coordinates:
  - $r^2 = x^2 + y^2 = 3^2 + 3^2 = 18$, so $r = \sqrt{18}$
  - $x = y$ implies $\theta = \frac{\pi}{4}$ and $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$
  - $z = 5$

Calculate Triple Integral Using Cylindrical Coordinates



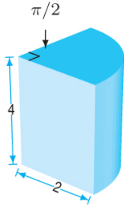
- Calculate $\int_R z\, dV$
- $R = \{0 \le r \le 4,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1\}$
- $dV = dx\, dy\, dz = dA\, dz = r\, dr\, d\theta\, dz$
- $$\begin{aligned} \int_R z\, dV &= \int_{z=0}^{z=1} \left(\int_{r=0}^{r=4} \left(\int_{\theta=0}^{\theta=2\pi} z\, d\theta \right) r\, dr \right) dz \\ &= \left(\int_{z=0}^{z=1} z\, dz \right) \left(\int_{r=0}^{r=4} r\, dr \right) \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) \\ &= \frac{1}{2} \left(\frac{16}{2} \right) 2\pi = 8\pi \end{aligned}$$

Example of Triple Integral Using Cylindrical Coordinates



- Let R be the shape shown above
- Assume the flat vertical sides are in the xz and yz planes and the base is in the xy plane
- In xyz coordinates, $R = \{x^2 + y^2 \leq 4,\ x, y \geq 0,\ 0 \leq z \leq 4\}$
- Calculate $\displaystyle\int_R xyz\, dV$

Example of Triple Integral Using Cylindrical Coordinates

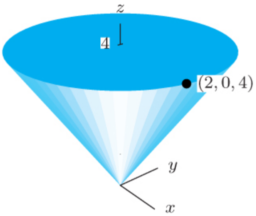


- $R = \{0 \leq r \leq 2,\ 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq z \leq 4\}$
- Therefore,

$$\begin{aligned}
\int_R xyz\, dV &= \int_{z=0}^{z=4} \left(\int_{r=0}^{r=2} \left(\int_{\theta=0}^{\theta=\frac{\pi}{2}} (r\cos\theta)(r\sin\theta) z\, d\theta \right) r\, dr \right)\ dz \\
&= \int_{z=0}^{z=4} z\, dz \int_{r=0}^{r=2} r^3\, dr \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \\
&= \frac{16}{4}\left(\frac{2^4}{4}\right) \int_{u=0}^{u=1} u\, du = 8,
\end{aligned}$$

where $u = \sin\theta$ and $du = \cos\theta\, d\theta$

Cone in Cylindrical Coordinates



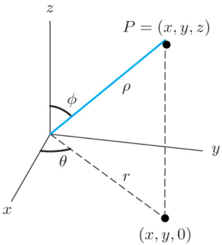
- Horizontal slice at height z is circular disk with radius proportional to z
- Thie means the region is given by $\sqrt{x^2+y^2} \leq cz$ for some constant c
- Since $(2,0,4)$ is on the boundary, equality holds

$$\sqrt{x^2+y^2} = 2 = cz = 4c \implies c = \frac{1}{2}$$

- Therefore, $R = \{\sqrt{x^2+y^2} \leq \frac{z}{2} \text{ and } 0 \leq z \leq 4\}$
- In cylindrical coordinates,

$$R = \left\{0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 4, 0 \leq r \leq \frac{z}{2}\right\}$$

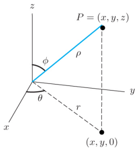
Spherical Coordinates in 3-Space



- Consider a point (x, y, z) with position vector $\vec{r} = \langle x, y, z\rangle$
- $\rho =$ distance from origin to $P = \sqrt{x^2 + y^2 + z^2}$
- $\phi =$ angle from $\vec{k}$ to $\vec{r}$
- $z = \vec{r} \cdot \vec{k} = |\vec{r}||\vec{k}| \cos\phi = \rho\cos\phi$
- $\theta =$ angle from $\vec{i}$ to $\langle x, y, 0\rangle$
- Same as polar coordinates in xy-plane
- $x = r\cos\theta$, $y = r\sin\theta$, where $r^2 = x^2 + y^2$
- Since $\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$,

$$r^2 = \sqrt{\rho^2 - z^2} = \sqrt{\rho^2(1 - (\cos\phi)^2)} = \rho\sin\phi$$

Spherical Coordinates in 3-Space



- Spherical coordinates consist of (ρ, ϕ, θ), where

$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

$$z = \rho\cos\phi, \qquad x = r\cos\theta = (\rho\sin\phi)\cos\theta$$

$$r = \rho\sin\phi, \qquad y = r\sin\theta = (\rho\sin\phi)\sin\theta$$

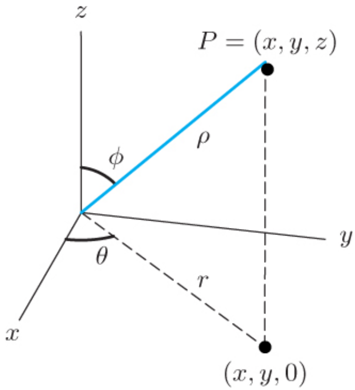
- $(x, y, z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)$
- Example: For the point $(1, 1, -\sqrt{2})$,

$$\rho = \sqrt{1+1+2} = 2$$

$$(z, r) = (-\sqrt{2}, \sqrt{2}) \implies \phi = \frac{3\pi}{4}$$

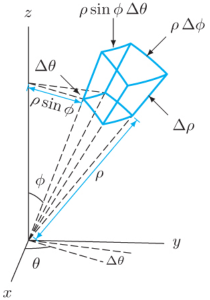
$$(x, y) = (1, 1) \implies \theta = \frac{\pi}{4}$$

IMPORTANT: Range of Values for Spherical Coordinates



- $\rho \geq 0$
- **REMEMBER THIS ONE:** $0 \leq \phi \leq \pi$
- $0 \leq \theta \leq 2\pi$ or $-\pi \leq \theta \leq \pi$

Integration Using Spherical Coordinates

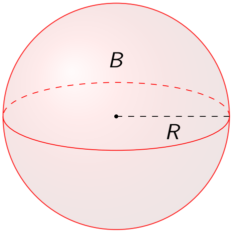


- $dV = (d\rho)(\rho\, d\phi)(\rho \sin\phi\, d\theta) = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$
- $0 \le \rho$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$ (or $-\pi \le \theta \le \pi$)
- Example: If B is the ball of radius R centered at the origin, then

$$\int_B f(x, y, z)\, dV$$

$$= \int_{\rho=0}^{\rho=R} \int_{\phi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, d\theta\, d\phi\, d\rho$$

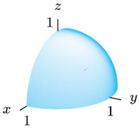
Volume of a Ball



- $B = \{0 \leq \rho \leq R,\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi\}$
- Volume of B is

$$\begin{aligned}
\int_B 1\, dV &= \int_{\rho=0}^{\rho=R} \int_{\pi=0}^{\phi=\pi} \int_{\theta=0}^{\theta=2\pi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \left(\int_{\rho=0}^{\rho=R} \rho^2 \, d\rho \right) \left(\int_{\phi=0}^{\phi=\pi} \sin\phi \, d\phi \right) \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) \\
&= \frac{2\pi R^3}{3} \left(- \cos\phi \big|_{\phi=0}^{\phi=\pi} \right) \\
&= \frac{4\pi R^3}{3}
\end{aligned}$$

Example of Integration Using Spherical Coordinates



- Region $Q = \{0 \le \rho \le 1,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \theta \le \frac{\pi}{2}\}$
- Suppose we want to calculate $\int_Q xyz,\ dV$

$$\int_Q xyz\, dV = \int_{\rho=0}^{\rho=1} \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho\cos\phi)\rho^2 \sin\phi \, d\theta\, d\phi\, d\rho$$

$$= \int_{\rho=0}^{\rho=1} \rho^5\, d\rho \int_{\phi=0}^{\phi=\frac{\pi}{2}} (\sin\phi)^3 \cos\phi\, d\phi \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos\theta \sin\theta\, d\theta$$

$$= \frac{1}{6} \int_{u=0}^{u=1} u^3\, du \int_{v=1}^{v=0} -v\, dv$$

$$= \frac{1}{6}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{48}$$