MATH-UA 123 Calculus 3: Double Integrals, Switching Order Of Integration

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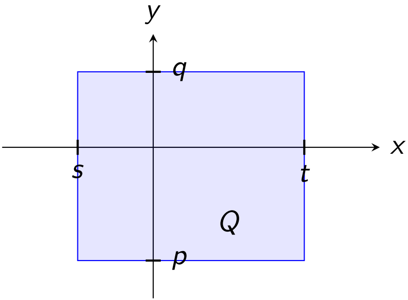
START RECORDING

Double Integral over a Rectangle



The double integral of a function $f(x, y)$ on the domain $Q = [p, q] \times [s, t]$ is

$$\int_Q f(x,y)\,dA = \int_{x=s}^{x=t} \left(\int_{y=p}^{y=q} f(x,y)\,dy \right) dx$$
$$= \int_{y=p}^{y=q} \left(\int_{x=s}^{x=t} f(x,y)\,dx \right) dy$$

Factoring a Double Integral

- Suppose $R = [a, b] \times [c, d]$
- Suppose the function $f(x, y)$ can be written as the product of a function of x only and a function of y only,

$$f(x, y) = p(x)q(y)$$

- Since a constant factor can be pulled out of an integral,

$$\begin{aligned}
\int_R p(x)q(y)\,dA &= \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} p(x)q(y)\,dy \right)\,dx \\
&= \int_{x=a}^{x=b} p(x) \left(\int_{y=c}^{y=d} q(y)\,dy \right)\,dx \\
&= \left(\int_{y=c}^{y=d} q(y)\,dy \right) \int_{x=a}^{x=b} p(x)\,dx
\end{aligned}$$

Example

Suppose $R = [-1, 2] \times [3, 7]$

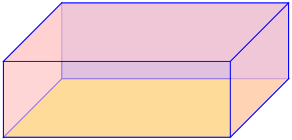
$$\int_R y^2 e^{-x}\, dA = \int_{x=-1}^{x=2} \left(\int_{y=3}^{y=7} y^2 e^{-x}\, dy \right) dx$$

$$= \int_{x=-1}^{x=2} e^{-x}\, dx \int_{y=3}^{y=7} y^2\, dy$$

$$= \left(-e^{-x} \Big|_{x=-1}^{x=2} \right) \left(\frac{y^3}{3} \Bigg|_{y=3}^{y=7} \right)$$

$$= (-e^{-2} - (-e^{1})) \left(\frac{7^3 - 3^3}{3} \right)$$

Triple Integral over a 3D Rectangular Box

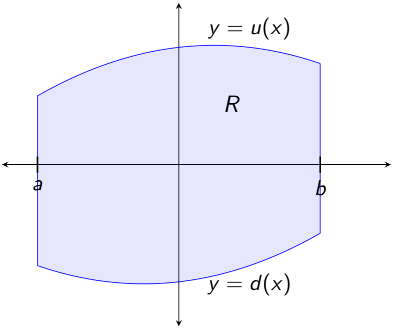


- The triple integral of a function $h(x, y, z)$ over a 3D rectangular region $R = [a, b] \times [c, d] \times [e, f]$ is

$$\int_R f(x, y, z)\, dV = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \left(\int_{z=e}^{z=f} h(x, y, z)\, dz \right) dy \right) dx$$

- The order of integration does not matter

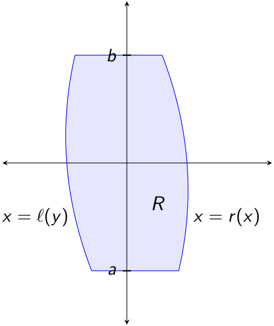
Double Integral Over Region Between Two Graphs



- Let R be the region bounded by the vertical lines $x = a$, $x = b$ and the graphs $y = d(x)$, $y = u(x)$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x, y)\, dA = \int_{x=a}^{x=b} \left(\int_{y=d(x)}^{y=u(x)} f(x, y)\, dy \right) dx$$

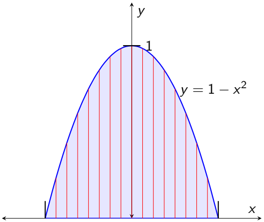
Double Integral Over Region Between Two Graphs



- Let R be the region bounded by the horizontal lines $y = a$, $y = b$ and the graphs $x = \ell(y)$, $x = r(y)$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{y=a}^{y=b} \left(\int_{x=\ell(y))}^{x=r(x)} f(x,y)\,dx \right) dy$$

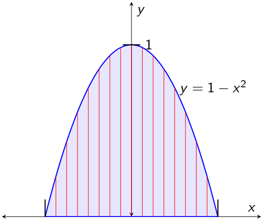
Double Integral Over Region Below Parabola



- The region R is the region between the graphs of $y = 0$ and $y = 1 - x^2$ with $-1 \leq x \leq 1$
- The integral of a function f over the region R is given by

$$\int_R f(x, y)\, dA = \int_{x=-1}^{x=1} \left(\int_{y=0}^{y=1-x^2} f(x, y)\, dy \right) dx$$

Double Integral Over Region Below Parabola



Example:

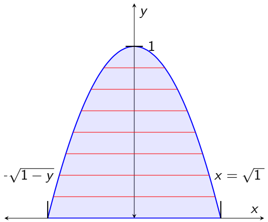
$$\int_R xy\, dA = \int_{x=-1}^{x=1} \left(\int_{y=0}^{y=1-x^2} x^2 y\, dy \right) dx$$

$$= \int_{x=-1}^{x=1} \left(\left[\frac{x^2 y^2}{2} \right]_{y=0}^{y=1-x^2} \right) dx = \frac{1}{2} \int_{x=-1}^{x=1} x^2 (1-x^2)^2\, dx$$

$$= \frac{1}{2} \int_{x=-1}^{x=1} x^2 - 2x^4 + x^6\, dx = \frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7} \Bigg|_{x=-1}^{x=1}$$

$$= 2 \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right)$$

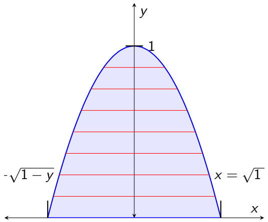
Double Integral Over Region Below Parabola



- R is the region bounded by the graphs $x = -\sqrt{1-y}$ and $x = \sqrt{1-y}$ with $0 \leq y \leq 1$
- The integral of a function $f(x,y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{y=0}^{y=1} \left(\int_{x=-\sqrt{1-y}}^{x=\sqrt{1-y}} f(x,y)\,dx \right) dy$$

Double Integral Over Region Below Parabola



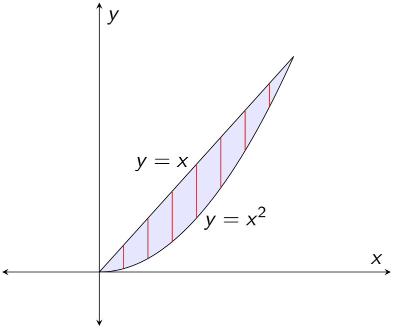
Example:

$$\int_R x^2 y\,dA = \int_{y=0}^{y=1} \left(\int_{x=-\sqrt{1-y}}^{x=\sqrt{1-y}} x^2 y\,dx \right) dy$$

$$= \int_{y=0}^{y=1} \left(\left[\frac{x^3 y}{3} \right]_{x=-\sqrt{1-y}}^{x=\sqrt{1-y}} \right) dy = \frac{2}{3} \int_{y=0}^{y=1} y(1-y)^{3/2}\,dy$$

$$= \frac{2}{3} \int_{u=1}^{u=0} (1-u)u^{3/2}\,du, \text{ where } u = 1 - y$$

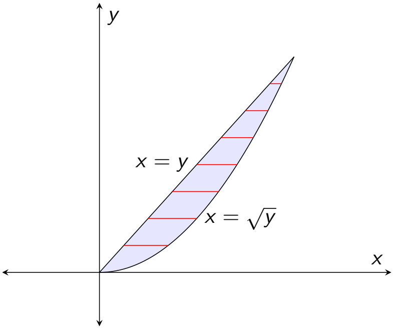
Double Integral Over Region Between Parabola and Line



- Let R be the region bounded by the graphs $y = x^2$ and $y = x$ with $0 \leq x \leq 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{x=0}^{x=1} \left(\int_{y=x^2}^{y=x} f(x,y)\,dy \right)\,dx$$

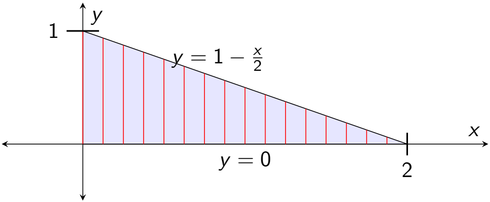
Double Integral Between Parabola and Line



- Let R be the region bounded by the graphs $x = \sqrt{y}$ and $x = y$ with $0 \leq y \leq 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\, dA = \int_{y=0}^{y=1} \left(\int_{x=y}^{x=\sqrt{y}} f(x,y)\, dx \right) dy$$

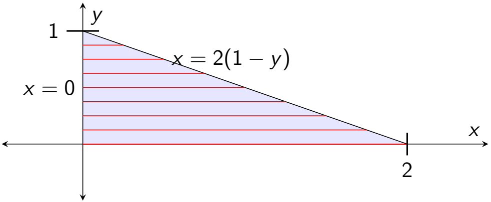
Double Integral over a Triangle



- Let R be the region bounded by the graph $y = 1 - \frac{x}{2}$ and the x-axis, where $0 \leq x \leq 2$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1-\frac{x}{2}} f(x,y)\,dy \right)\,dx$$

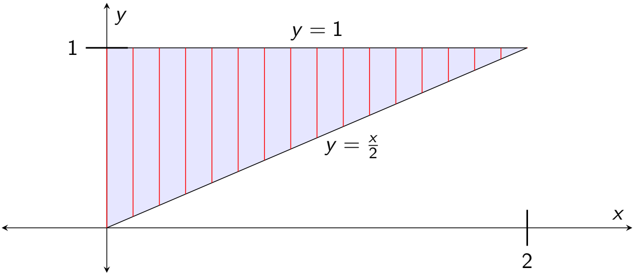
Double Integral over a Triangle



- Let R be the region bounded by the graph $x = 2(1 - y)$ and the y-axis, where $0 \leq y \leq 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x, y)\, dA = \int_{y=0}^{y=1} \left(\int_{x=0}^{x=2(1-y)} f(x, y)\, dy \right) dx$$

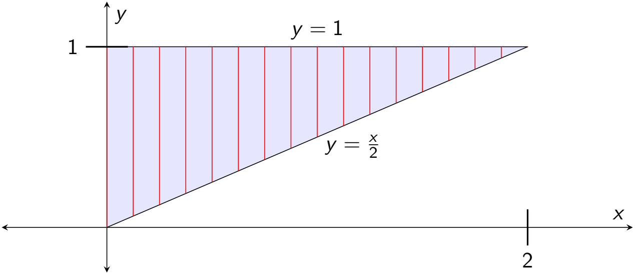
Double Integral over a Triangle



- Let R be the region bounded by the graphs $y = \frac{x}{2}$ and $y = 1$, where $0 \leq x \leq 2$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\, dA = \int_{x=0}^{x=2} \left(\int_{y=\frac{x}{2}}^{y=1} f(x,y)\, dy \right) dx$$

Double Integral over a Triangle



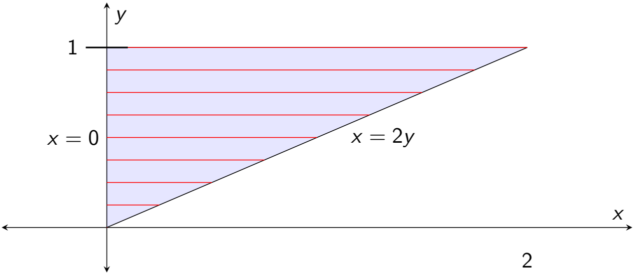
Example:

$$\int_R 8xy\, dA = \int_{x=0}^{x=2} \left(\int_{y=\frac{x}{2}}^{y=1} 8xy\, dy \right) dx = \int_{x=0}^{x=2} \left(4xy^2 \Big|_{y=\frac{x}{2}}^{y=1} \right) dx$$

$$= \int_{x=0}^{x=2} 4x - x^3\, dx = 2x^2 - \frac{x^4}{4} \Bigg|_{x=0}^{x=2}$$

$$= 8 - \frac{16}{4} = 4$$

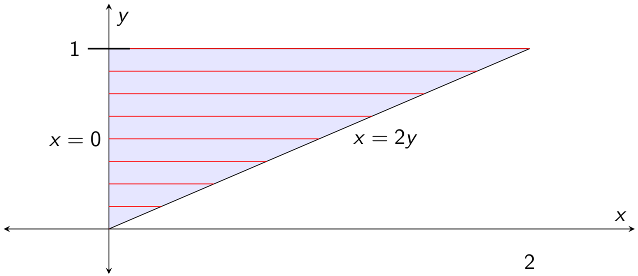
Double Integral over a Triangle



- Let R be the region bounded by the graphs $x = 0$ and $x = 2y$, where $0 \leq y \leq 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{y=0}^{y=1} \left(\int_{x=0}^{x=2y} f(x,y)\,dx \right) dy$$

Double Integral over a Triangle



Example:

$$\begin{aligned}\int_R 8xy\,dA &= \int_{y=0}^{y=1}\left(\int_{x=0}^{x=2y} 8xy\,dx\right)\,dy \\ &= \int_{y=0}^{y=1}\left(4x^2y\Big|_{x=0}^{x=2y}\right)\,dy \\ &= \int_{y=0}^{y=1} 16y^3\,dy = 4y^4\Big|_{y=0}^{y=1} = 4\end{aligned}$$

Switching Order Of Integration

- Suppose we want to integrate

$$\int_{x=0}^{x=6} \left(\int_{y=x/3}^{y=2} x\sqrt{y^3+1}\, dy \right) dx$$

- There is no way to find the antiderivative of the function $\sqrt{1+y^3}$
- Instead, try switching the order of integration
- **Endpoints will change!**

$$\int_{x=0}^{x=6} \left(\int_{y=x/3}^{y=2} x\sqrt{y^3+1}\, dy \right) dx$$

$$= \int_{y=?}^{y=?} \left(\int_{x=?}^{x=?} x\sqrt{y^3+1}\, dx \right) dy$$

Switching Order Of Integration

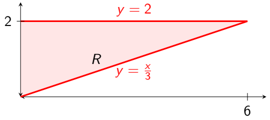
- We want to integrate

$$\int_{x=0}^{x=6} \left(\int_{y=x/3}^{y=2} x\sqrt{y^3+1}\, dy \right) \, dx$$

- To switch order of integration, find the domain R such that

$$\int_{x=0}^{x=6} \left(\int_{y=x/3}^{y=2} x\sqrt{y^3+1}\, dy \right) \, dx = \int_R x\sqrt{y^3+1}\, dA$$

- According to the left side, R is the region between the graphs $y = \frac{x}{3}$ and $y = 2$ with $0 \le x \le 6$

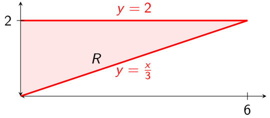


Switching Order Of Integration

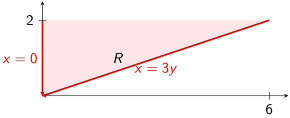
- We found that

$$\int_{x=0}^{x=6}\left(\int_{y=x/3}^{y=2} x\sqrt{y^3+1}\,dy\right)\,dx = \int_R x\sqrt{y^3+1}\,dA,$$

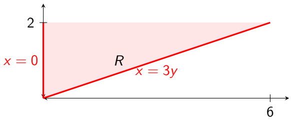
where R is the region between the graphs $y=\frac{x}{3}$ and $y=2$ with $0\le x\le 6$



- R is also the region between the graphs $x=0$ and $x=3y$ with $0\le y\le 2$



Switching Order Of Integration



$$\int_{x=0}^{x=6} \left(\int_{y=x/3}^{y=2} x\sqrt{y^3+1}\, dy \right)\, dx = \int_R x\sqrt{y^3+1}\, dA$$
$$= \int_{y=0}^{y=2} \left(\int_{x=0}^{x=3y} x\sqrt{y^3+1}\, dx \right)\, dy$$

Calculation of New Integral Using Substitution

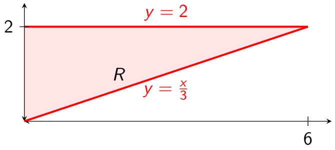
$$\begin{aligned}
\int_{y=0}^{y=2}\left(\int_{x=0}^{x=3y} x\sqrt{y^3+1}\,dx\right)\,dy &= \int_{y=0}^{y=2} \sqrt{y^3+1}\left(\left.\frac{x^2}{2}\right|_{x=0}^{x=3y}\right)\,dy \\
&= \frac{9}{2}\int_{y=0}^{y=2} \sqrt{y^3+1}\,y^2\,dy \\
&= \frac{9}{2}\int_{u=1}^{u=9} u^{1/2}\,\frac{du}{3} \\
&= \frac{3}{2}\left.\frac{u^{3/2}}{3/2}\right|_{u=1}^{u=9} \\
&= 9^{3/2} - 1^{3/2} = 26
\end{aligned}$$

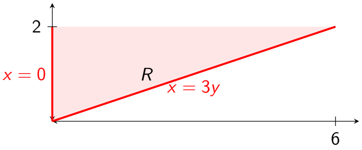
where

$$u = y^3 + 1, \qquad du = 3y^2\,dy,$$

$$u = 1 \text{ when } y = 0, \qquad u = 9 \text{ when } y = 2$$

Switching Order Of Integration





$$\int_{x=0}^{x=6}\left(\int_{y=x/3}^{y=2} x\sqrt{y^3+1}\,dy\right)\,dx = \int_R x\sqrt{y^3+1}\,dA$$

$$= \int_{y=0}^{y=2}\left(\int_{x=0}^{x=3y} x\sqrt{y^3+1}\,dx\right)\,dy$$

$$= 26$$