MATH-UA 123 Calculus 3: Double and Triple Integrals

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Single Integral over an Interval

The integral

$$\int_{x=a}^{x=b} f(x)\,dx,$$

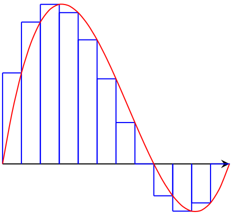
where f is a function whose domain contains the interval $[a, b]$ has the following interpretations:

- Signed area of region between the graph of f and the x-axis from $x = a$ to $x = b$
  - If $a \leq b$ and $f \geq 0$, then integral ≥ 0
  - If $a \leq b$ and $f \leq 0$, then integral ≤ 0
  - If $a \geq b$ and $f \geq 0$, then integral ≤ 0
  - If $a \geq b$ and $f \leq 0$, then integral ≥ 0
- If f is nonnegative, it can be viewed as a mass density function of a thin rod. The integral is the total mass of the rod.
- Average value of $f(x)$, $x \in [a, b]$ is

$$A = \frac{1}{b-a} \int_{x=a}^{x=b} f(x)\,dx$$

  The signed area of the rectangle with height A and width $b - a$ is equal to the integral

Definition of an Integral Over Interval

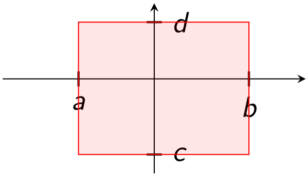


Recall that the integral of a continuous function $f(x)$ over an interval $[a, b] = \{a \leq x \leq b\}$ is defined to be the limit of signed areas of rectangles

$$\int_{x=a}^{x=b} f(x)\,dx = \lim_{N\to\infty} \frac{b-a}{N}(f(x_1) + f(x_2) + \cdots + f(x_N)),$$

where $x_k = a + \frac{k}{N}(b-a)$, for each $k = 1, \ldots, N$.

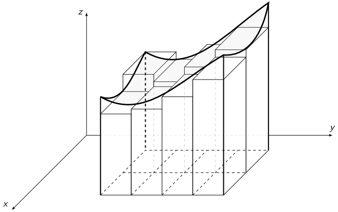
Rectangle in 2-Space



- Notation: Recall that an interval is written as $[a, b] = \{a \leq x \leq b\}$
- $[a, b] \times [c, d]$ is a rectangle in 2-space,

$$[a, b] \times [c, d] = \{(x, y) \ : \ a \leq x \leq b \text{ and } c \leq y \leq d\}$$

Definition of Double Integral Over Rectangle



The integral of a continuous function $f(x, y)$ over a rectangle

$$R = [a, b] \times [c, d]$$

is defined to be the limit of the sum of signed volumes of rectangular boxes,

$$\int_R f(x, y)\, dA = \lim_{M \to \infty} \lim_{N \to \infty} \sum_{j=1}^{M} \sum_{k=1}^{N} \left(\frac{b-a}{M} \right) \left(\frac{d-c}{N} \right) f(x_j, y_k),$$

where

$$x_j = a + j \left(\frac{b-a}{N} \right) \text{ and } y_k = c + k \left(\frac{d-c}{N} \right)$$

Interpretations of a Double Integral over a Rectangle

The double integral

$$\int_R f(x,y)\,dx\,dy,$$

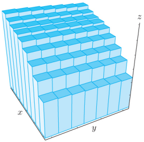
where the domain of f contains the rectangle $R = [a,b] \times [c,d]$, has the following interpretations:

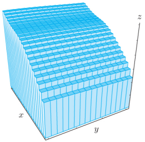
- Signed volume of region between the graph of f and the rectangle R in the xy-plane
- If f is nonnegative, it can be viewed as a mass density function of a thin rectangular sheet. The integral is the total mass of the sheet.
- Average value of $f(x,y)$, $(x,y) \in [a,b]$ is

  $$A = \frac{1}{b-a}\int_{x=a}^{x=b} f(x)\,dx$$

  The signed volume of the rectangular box with height A and base $[a,b] \times [c,d]$ is equal to the integral

Formula for Double Integral Over Rectangle





$$\int_R f(x,y)\,dA = \lim_{M\to\infty}\lim_{N\to\infty}\sum_{j=1}^{M}\sum_{k=1}^{N}\left(\frac{b-a}{M}\right)\left(\frac{d-c}{N}\right)f(x_j,y_k)$$

$$= \lim_{M\to\infty}\left(\frac{b-a}{M}\right)\sum_{k=1}^{M}\left(\lim_{N\to\infty}\left(\frac{d-c}{N}\right)\sum_{k=1}^{N}f(x_j,y_k)\right)$$

$$= \lim_{M\to\infty}\left(\frac{b-a}{M}\right)\sum_{k=1}^{M}\int_{y=c}^{y=d} f(x_j,y)\,dy$$

$$= \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)\,dy\right)\,dx$$

Order of Integration Does Not Matter

- If f is continuous, then the same calculation works with x and y switched
- Therefore,

$$\int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \, dy \right) \, dx = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} \, dx \right) \, dy$$

- So we just write this double integral as

$$\int_R f(x, y) \, dA$$

- $dA = dx \, dy$ means roughly the area of a small rectangle
- So

$$\int_R f(x, y) \, dA = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \, dy \right) \, dx = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} \, dx \right) \, dy$$

Examples of Double Integrals Over a Rectangle

- The integral
$$\int_{[-1,2]\times[3,4]} 5\,dA$$
is the volume of a box whose base is the rectangle $[-1,2]\times[3,4]$ andheight is 5
- Therefore,
$$\int_{[-1,2]\times[3,4]} 5\,dA = (2--1)(4-3)5 = 15$$
- On the other hand,
$$\begin{aligned}\int_{[-1,2]\times[3,4]} x\,dA &= \int_{y=3}^{y=4}\left(\int_{x=-1}^{x=2} x\,dx\right)\,dy\\ &= \int_{y=3}^{y=4} \frac{x^2}{2}\bigg|_{x=-1}^{x=2}\,dy\\ &= \int_{y=3}^{y=4} 2-\left(-\frac{1}{2}\right)\,dy = \frac{5}{2}\end{aligned}$$

Example of Double Integral Over a Rectangle

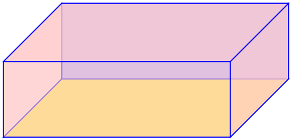
$$\begin{aligned}
\int_{[-1,2]\times[3,4]} ye^{-x}\,dA &= \int_{y=3}^{y=4} \left(\int_{x=-1}^{x=2} ye^{-x}\,dx \right) dy \\
&= \int_{y=3}^{y=4} y \left(\int_{x=-1}^{x=2} e^{-x}\,dx \right) dy \\
&= \left(\int_{y=3}^{y=4} y\,dy \right) \left(\int_{x=-1}^{x=2} e^{-x}\,dx \right) \\
&= \frac{y^2}{2} \Bigg|_{y=3}^{y=4} -e^{-x} \Big|_{x=-1}^{x=2} \\
&= \frac{16-9}{4} + (-e^{-2} - (-e^{1})) = \frac{7}{4} + e - e^{-2}
\end{aligned}$$

Factoring a Double Integral

- Suppose $R = [a, b] \times [c, d]$
- Since a constant factor can be pulled out of an integral,

$$\begin{aligned}\int_R f(x)g(y)\,dA &= \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x)g(y)\,dy \right)\,dx \\ &= \int_{x=a}^{x=b} f(x) \left(\int_{y=c}^{y=d} g(y)\,dy \right)\,dx \\ &= \left(\int_{y=c}^{y=d} g(y)\,dy \right) \left(\int_{x=a}^{x=b} f(x)\,dx \right)\end{aligned}$$

Triple Integral over a 3D Rectangular Box



- Given a rectangular region $R = [a, b] \times [c, d] \times [g, h]$ and a function $f(x, y, z)$, define

$$\int_R f(x, y, z)\, dV = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \left(\int_{z=g}^{z=h} f(x, y, z)\, dz \right) dy \right) dx$$

- Order of integration does not matter
- Calculation of a triple integral over a 3D rectangular box is exactly the same as calculating a double integral over a rectangle

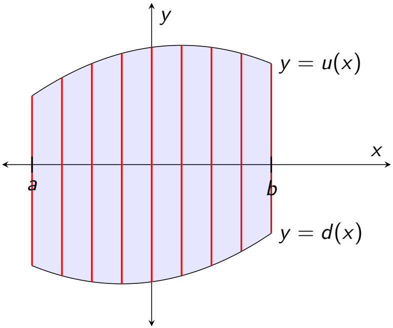
Interpretations of a Triple Integral Over a 3D Rectangular Box

- Signed volume of 4-dimentional region between the graph of f and the xyz-plane
- If f is nonnegative, then it can be viewed as the mass density of a block and the integral is the total mass
- Average value of f over the region R is

$$A = \frac{1}{V(R)} \int_R f(x, y, z)\, dA,$$

where $V(R)$ is the volume of R, $V(R) = (b-a)(c-d)(h-g)$

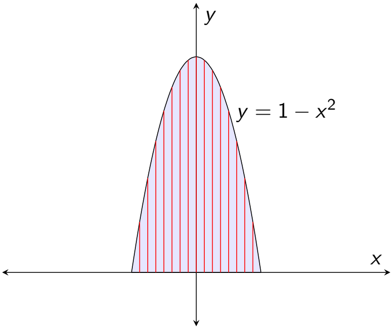
Double Integral Over Region Between Two Graphs



- Let R be the region bounded by the vertical lines $x = a$, $x = b$ and the graphs $y = d(x)$ and $y = u(x)$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{x=a}^{x=b} \left(\int_{y=d(x)}^{y=u(x)} f(x,y)\,dy \right)\,dx$$

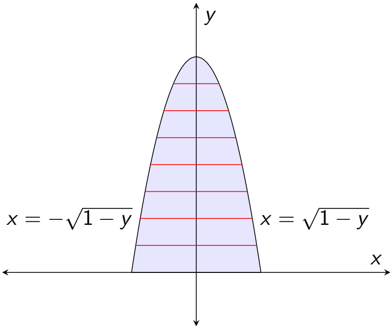
Double Integral Over Region Between Two Graphs



- Let R be the region bounded by the graph $y = 1 - x^2$ and the axis, where $-1 \leq x \leq 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x, y)\, dA = \int_{x=-1}^{x=1} \left(\int_{y=0}^{y=1-x^2} f(x, y)\, dy \right) dx$$

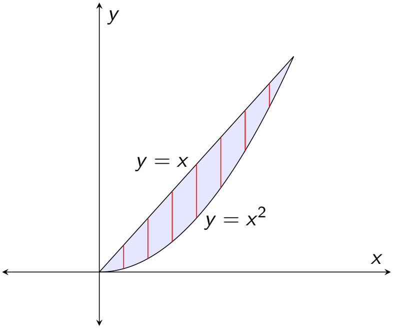
Double Integral Over Region Between Two Graphs



- Let R be the region bounded by the graphs $x = -\sqrt{1-y}$ and $x = \sqrt{1-y}$, where $0 \leq y \leq 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{y=0}^{y=1} \left(\int_{x=-\sqrt{1-y}}^{x=\sqrt{1-y}} f(x,y)\,dx \right) dy$$

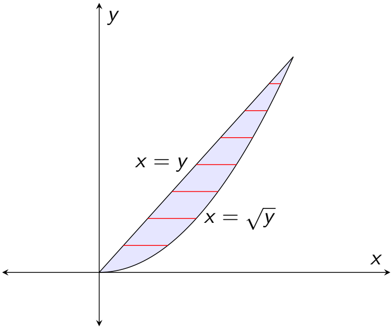
Double Integral Over Region Between Two Graphs



- Let R be the region bounded by the graphs $y = x^2$ and $y = x$, where $0 \leq x \leq 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{x=0}^{x=1} \left(\int_{y=x^2}^{y=x} f(x,y)\,dy \right)\,dx$$

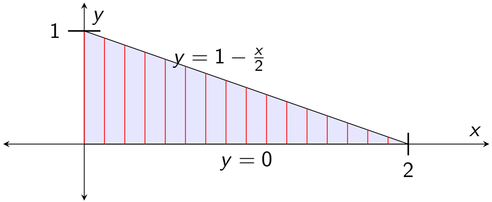
Double Integral over a Non-Rectangular Region



- Let R be the region bounded by the graphs $x = \sqrt{y}$ and $x = y$, where $0 \leq y \leq 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\, dA = \int_{y=0}^{y=1} \left(\int_{x=y}^{x=\sqrt{y}} f(x,y)\, dx \right) dy$$

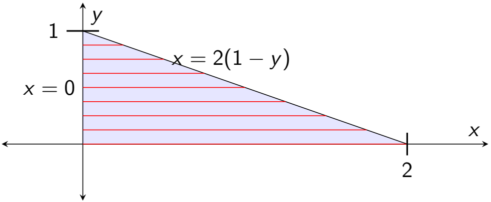
Double Integral over a Triangle



- Let R be the region bounded by the graph $y = 1 - \frac{x}{2}$ and the x-axis, where $0 \leq x \leq 2$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x, y)\, dA = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1-\frac{x}{2}} f(x, y)\, dy \right) dx$$

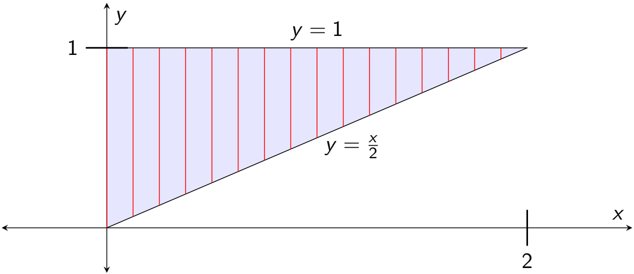
Double Integral over a Triangle



- Let R be the region bounded by the graph $x = 2(1 - y)$ and the y-axis, where $0 \leq y \leq 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{y=0}^{y=1} \left(\int_{x=0}^{x=2(1-y)} f(x,y)\,dy \right)\,dx$$

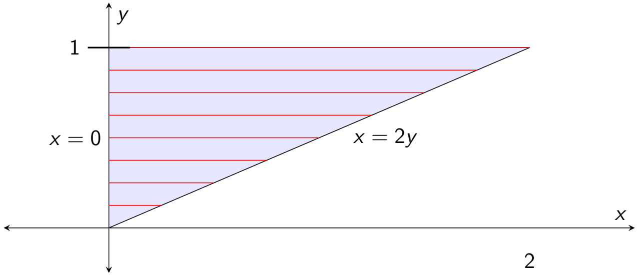
Double Integral over a Triangle



- Let R be the region bounded by the graphs $y = \frac{x}{2}$ and $y = 1$, where $0 \leq x \leq 2$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{x=0}^{x=2} \left(\int_{y=\frac{x}{2}}^{y=1} f(x,y)\,dy \right) dx$$

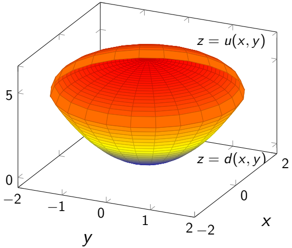
Double Integral over a Triangle



- Let R be the region bounded by the graphs $x = 0$ and $x = 2y$, where $0 \le y \le 1$
- The integral of a function $f(x, y)$ over the region R is equal to

$$\int_R f(x,y)\,dA = \int_{y=0}^{y=1} \left(\int_{x=0}^{x=2y} f(x,y)\,dx \right) dy$$

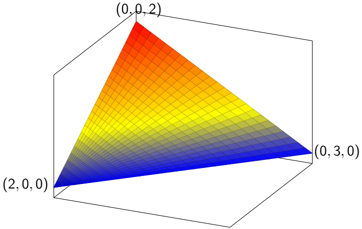
Triple Integral Over Region Between Two Graphs



- $D =$ domain in xy-plane
- $d(x,y)$ and $u(x,y)$ functions such that $d(x,y) \leq u(x,y)$, for all $(x,y) \in D$
- $R = \{(x,y,z) \ : \ d(x,y) \leq z \leq u(x,y)\}$
- The integral of $f(x,y,z)$ over the region R is equal to

$$\int_R f(x,y,z)\,dV = \int_{(x,y)\in D} \left(\int_{z=d(x,y)}^{z=u(x,y)} f(x,y,z)\,dz \right) dA$$

Triple Integral Between Two Graphs Over a Triangle



- $R =$ region above $z = 0$ and below $3x + 2y + 3z = 6$
- $T =$ triangle in xy-plane with vertices $(0, 0)$, $(2, 0)$, $(0, 3)$
- The integral of $f(x, y, z)$ over the region R is equal to

$$\int_R f(x,y,z)\,dV = \int_{(x,y)\in T} \left(\int_{z=0}^{z=\frac{6-3x-2y}{3}} f(x,y,z)\,dz \right) dA$$

$$= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=3-\frac{3}{2}x} \left(\int_{z=0}^{z=\frac{6-3x-2y}{3}} f(x,y,z)\,dz \right) dy$$