MATH-UA 123 Calculus 3: Global Optimization, Lagrange Multipliers

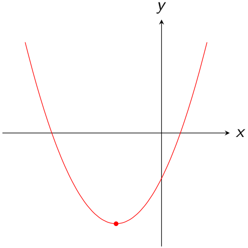
Deane Yang

Courant Institute of Mathematical Sciences
New York University

October 18, 2021

START RECORDING

Global Optimization on the Real Line



- Suppose $f(x)$ is a smooth function on the entire real line
- Optimal values, if they exist, must occur at a critical point
- To find optima:
  - Study what happens when $x \to \pm\infty$
  - Find all critical points and calculate f at each of them
- In picture:
  - $f(x) \to +\infty$ as $x \to \pm\infty$, which implies that f has no maximum value
  - f is bounded from below, which means that it has a minimum value
  - There is only one critical point, so that has to be the minimum

Global Optimization in 2-Space

- Find rectangular cardboard box without a top that encloses a given volume V but using the minimum amount of cardboard
- If dimensions of box are H by W by D, then

$$\text{Volume } V = HWD$$
$$\text{Area of card board } A = 2(HW + HD) + WD$$

- V is constant, and we want to minimize A
- Elminiate one variable $H = \frac{V}{WD}$:

$$A(W, D) = 2\frac{V}{WD}(W + D) + WD = 2V(\frac{1}{D} + \frac{1}{W}) + WD$$

Optimal Cardboard Box

- $A(W, D) = 2V(\frac{1}{D} + \frac{1}{W}) + WD$
- Solution must be at a critical point of A
- Find critical points:

$$A_W = -\frac{2V}{W^2} + D = 0, \qquad A_D = -\frac{2V}{D^2} + W = 0$$

$$D = \frac{2V}{W^2}, \qquad W = \frac{2V}{D^2} = 2V\frac{W^4}{4V^2} = \frac{W^4}{2V}$$

- Therefore,

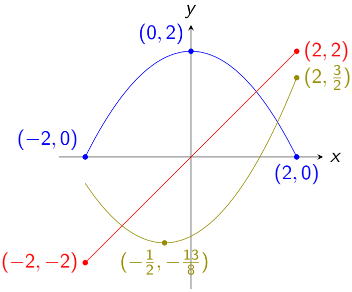
$$0 = \frac{W^4}{2V} - W = W\left(\frac{W^3}{2V} - 1\right)$$

- Since $W \neq 0$,

$$W = (2V)^{1/3} \qquad D = \frac{2V}{W^2} = (2V)^{1/3}$$

$$H = \frac{V}{WD} = \frac{V}{(2V)^{2/3}} = 2(2V)^{1/3}$$

Global Optimization on a Bounded Interval



- The global optima of a smooth function on a bounded closed interval are always at critical or end points
- Here, we have three functions:

$$f(x) = 2 - \frac{1}{2}x^2$$

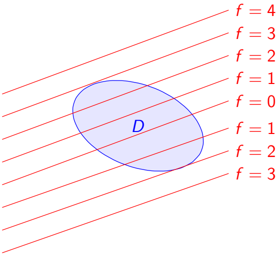
$$g(x) = x$$

$$h(x) = \frac{1}{2}(x^2 - x - 3)$$

Finding Optimal Values and Points on an Interval

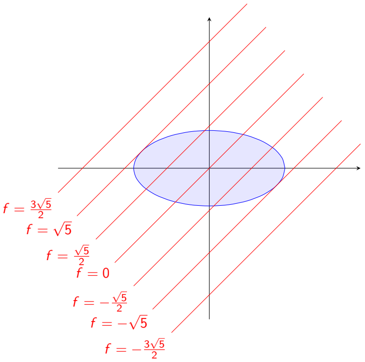
- Find all of the critical points that lie in the interval
- Calculate the value of the function at each critical and each end point
- Identify where the function is maximum and where it is minimum

Global Optima on a Bounded Domain in 2-Space



- Suppose $D = \{(x, y) \ : \ g(x, y) \leq 1\}$
- Maximize or minimize $f(x, y)$ with (x, y) restricted to the domain D
- An optimal point must be either a critical point or a point on the boundary
- If optimal point is on bondary, then it must be at a point where the contour of f and the boundary are tangent
  - Where $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$ for some scalar λ

Example



- Optimize $f(x, y) = y - x$ over all (x, y) such that $\frac{x^2}{4} + y^2 \leq 1$
- Since $\vec{\nabla} f = \langle -1, 1 \rangle$, there are no critical points
- The boundary is the contour $g = 1$, where $g(x, y) = \frac{x^2}{4} + y^2$
- Solve for x, y, λ such that

$$\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y) \text{ and } g(x, y) = 1$$

Constrained Optimization Example

- Constraint: $g = 1$, where $g(x, y) = \frac{x^2}{4} + y^2$
- Objective function: $f(x, y) = y - x$
- Solve for (x, y) and λ such that $\vec{\nabla} f = \lambda \vec{\nabla} g$

$$\langle -1, 1 \rangle = \lambda \langle \frac{x}{2}, 2y \rangle$$

- $\lambda \neq 0$ because left side is nonzero
- Therefore,

$$\langle -\lambda^{-1}, \lambda^{-1} \rangle = \langle \frac{x}{2}, 2y \rangle$$

$$2y = -\frac{x}{2}$$

$$y = -\frac{x}{4}$$

$$1 = \frac{x^2}{4} + \frac{x^2}{16} = \frac{5}{16} x^2$$

$$x = \pm \frac{4}{\sqrt{5}}$$

Constrained Optimization Example

- Constraint: $g = 1$, where $g(x, y) = \frac{x^2}{4} + y^2$
- Objective function: $f(x, y) = y - x$
- Solve for (x, y) and λ such that $\vec{\nabla} f = \lambda \vec{\nabla} g$
- $y = -\frac{x}{4}$ and $x = \pm \frac{4}{\sqrt{5}}$
- Therefore,

$$(x, y) = (\frac{4}{\sqrt{5}}, -\frac{1}{\sqrt{5}}) \text{ or } (-\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}})$$

- Calculate values of f

$$f(\frac{4}{\sqrt{5}}, -\frac{1}{\sqrt{5}}) = -\sqrt{5} \text{ and } f(-\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}}) = \sqrt{5}$$

- The constrained maximum value of f is $\sqrt{5}$ and occurs at $(x, y) = (-\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}})$
- The constrained minimum value of f is $-\sqrt{5}$ and occurs at $(x, y) = (\frac{-4}{\sqrt{5}}, -\frac{1}{\sqrt{5}})$

Optimization on a Bounded Domain

- Suppose you want to find the maximum or minimum value of a function f on a closed bounded domain D in 2-space
- Closed means D contains its boundary
- The maximum and minimum points of f must either be critical points in D or lie on the boundary of D
- To find the optimal points and corresponding values of f:
  - Find all critical points of f that lie in D
  - Find all maximum or minimum points on the boundary D by doing constrained optimization
  - Calculate the value of f on each point identified in previous steps

Constrained Optimization on a Contour

- Objective function $f(x, y)$
- Constraint equation $g(x, y) = c$, where c is a constant
- Assume
  - The contour $g = c$ is bounded
  - $\vec{\nabla} g(x, y) \neq 0$ for any (x, y) in the contour $g = c$
- The constrained maxima and minima must occur at points in the contour that are either critical points of f or where $\vec{\nabla} f$ and $\vec{\nabla} g$ point in the same or opposite directions, i.e.

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

- Note that $\lambda = 0$ corresponds to a critical point of f
- Solution process:
  - Find all points (x, y) such that $g(x, y) = 0$ and there is a scalar λ such that $\vec{\nabla} f(x, y) = \lambda \vec{\nabla} g(x, y)$
  - Calculate f at all points found in previous step
  - Identify maximum or minimun points and values

Two Approaches to solving a Lagrange Multiplier Problem

- **Approach 1**
  - Use each of first two equations to solve for λ
  - Combine two equations from first step to get an equation in x and y only
  - Use equation from previous step and the third (constraint) equation to solve for x and y
- **Approach 2 (use only if first step is relatively easy to do)**
  - Use first two equations to get formulas for x and y in terms of λ only
  - Substitute formulas from previous step into the third (constraint) equation
  - Solve resulting equation for λ
  - Substitute formula for λ into the formulas for x and y

Example

- Objective function $f(x, y) = (x + 2)^2 + (y - 1)^2$
- Domain: $x^2 + y^2 \leq 1$
- Critical points of f:
  - $\vec{\nabla} f = 2\langle x + 2, y - 1\rangle$
  - Only one: $(x, y) = (-2, 1)$
  - Check if it is in D: $x^2 + y^2 = 4 + 1 = 5 > 1$
  - Not in D, so remove from consideration
- Constrained optimization on boundary

Comnstrained Optimization on Boundary

- Objective function: $f(x, y) = (x + 2)^2 + (y - 1)^2$
- Constraint equation: $x^2 + y^2 = 1$
- Constraint is $g = 1$, where $g(x, y) = x^2 + y^1$
- $\vec{\nabla} g = 2\langle x, y\rangle$
- $\vec{\nabla} f = \lambda \vec{\nabla} g$ and $g = 1$ imply

$$\begin{aligned} x + 2 &= \lambda x \\ y - 1 &= \lambda y \\ x^2 + y^2 &= 1 \end{aligned}$$

Approach 1

- Solve first two equations for λ:

$$\frac{x+2}{x} = \lambda = \frac{y-1}{y}$$
$$1 + \frac{2}{x} = 1 - \frac{1}{y}$$
$$y = -\frac{x}{2}$$

- If $x = 0$, then there is no solution to first equation
- If $y = 0$, then there is no solution to second equation
- Assume x and y are nonzero and substitute equation above into constraint equation

$$1 = x^2 + y^2 = x^2 + \frac{x^2}{4} = \frac{5}{4}x^2$$

- Therefore, $x = \pm\frac{2}{\sqrt{5}}$ and $y = -\frac{x}{2}$
- The solutions are $(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ and $(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}})$

Approach 2

- Equations are

$$x + 2 = \lambda x$$
$$y - 1 = \lambda y$$
$$x^2 + y^2 = 1$$

- Solve first two equations for x and y in terms of λ

$$x = \frac{2}{\lambda - 1} \text{ and } y = -\frac{1}{\lambda - 1}$$

  If $\lambda = 1$, there is no solution
- Assume $\lambda \neq 1$
- Substitute equations above into third equation and solve for λ

$$1 = \frac{4}{(\lambda - 1)^2} + \frac{1}{(\lambda - 1)^2} = \frac{5}{(\lambda - 1)^2} \implies \lambda = 1 \pm \frac{1}{\sqrt{5}}$$

- Substitute into formulas for x and y

$$(x, y) = (\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}) \text{ or } (x, y) = (-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$$

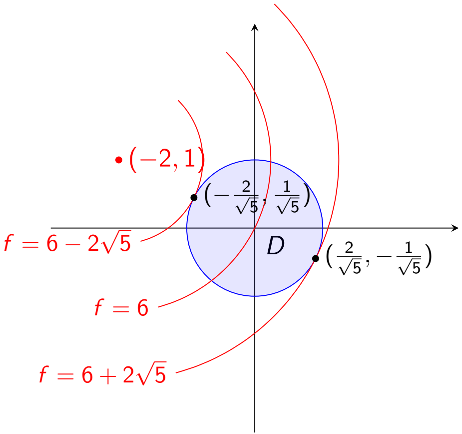
Calculate Values of Objective Function

- Calculate f:

$$f(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}) = \left(-\frac{2}{\sqrt{5}} + 2\right)^2 + \left(\frac{1}{\sqrt{5}} - 1\right)^2$$
$$= 6 - 2\sqrt{5}$$
$$f(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}) = \left(\frac{2}{\sqrt{5}} + 2\right)^2 + \left(-\frac{1}{\sqrt{5}} - 1\right)^2$$
$$= 6 + 2\sqrt{5}$$

- Constrained maximum value of $6 + 2\sqrt{5}$ occurs at $(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}})$
- Constrained minimum value of $6 - 2\sqrt{5}$ occurs at $(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$

Picture of Optimization Problem and Solution



- $D = \{x^2 + y^2 = 1\}$
- $f(x, y) = (x + 2)^2 + (y - 1)^2$