MATH-UA 123 Calculus 3: Critical Points, Optimization

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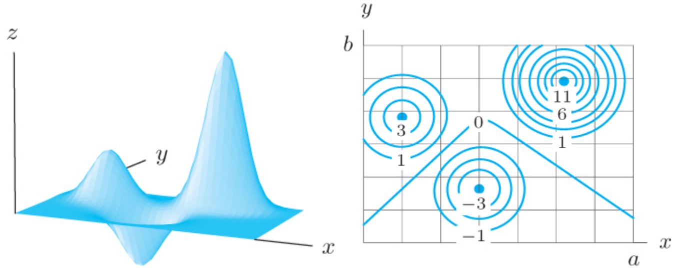
START RECORDING

Shape Versus Contours of a Graph

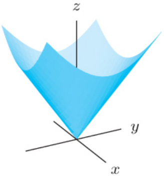


- Key features of a surface:
  - Peaks
  - Bottoms
  - Ridges and valleys between peaks or bottoms
- There are at least 4 points where the gradient is zero
  - Two peaks
  - One bottom
  - One point in between the peaks and bottom, where the contour consists of two intersecting curves

Some Possible Shapes of a Graph

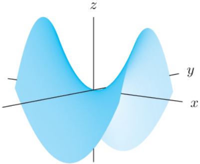
Isolated local maximum

Isolated local minimum

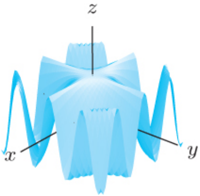


Isolated local minimum

Line of local minima



Saddle point

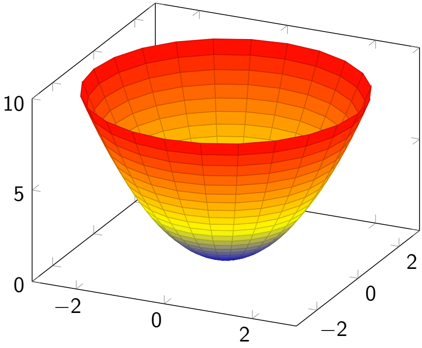


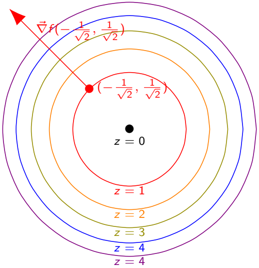
Complicated set of local maxima

Critical Point of Function

- A point (x_0, y_0) is a critical point of a function f if
  - It is in the domain of f
  - The gradient $\vec{\nabla} f(x_0, y_0, z_0)$ is either zero or undefined
- Possible shapes of a surface near a critical point
  - Isolated local maximum: Top of a hill
  - Isolated local minimum: Bottom of a bowl
  - Curve of local minima: Bottom of a valley
  - Curve of local maxima: Top of a ridge
  - Saddle point
  - Other

Circular Paraboloid

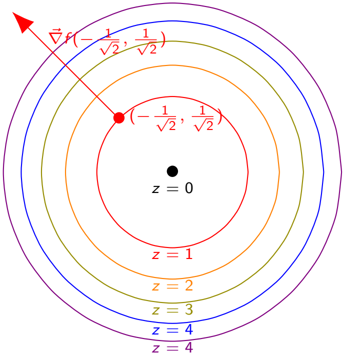


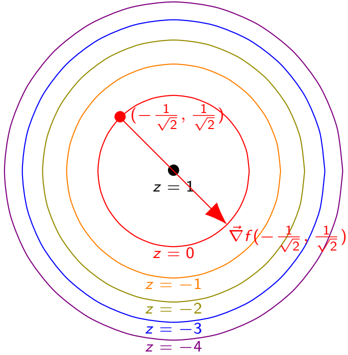


$$f(x, y) = x^2 + y^2$$
$$\vec{\nabla} f(x, y) = 2\langle x, y\rangle$$

Local minimum

Contours of Circular Paraboloids





$$f(x, y) = x^2 + y^2$$

$$\vec{\nabla} f(x, y) = 2\langle x, y\rangle$$

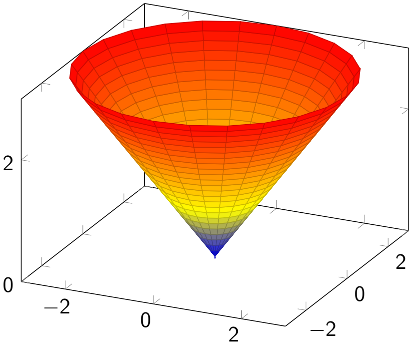
Local minimum

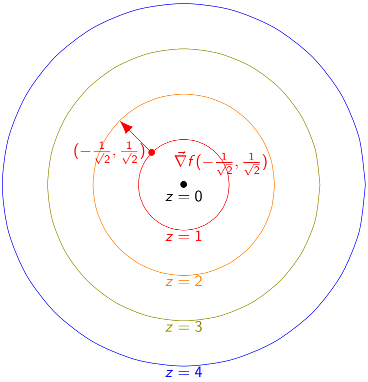
$$f(x, y) = 1 - x^2 - y^2$$

$$\vec{\nabla} f(x, y) = -2\langle x, y\rangle$$

Local maximum

Contour of Circular Cone



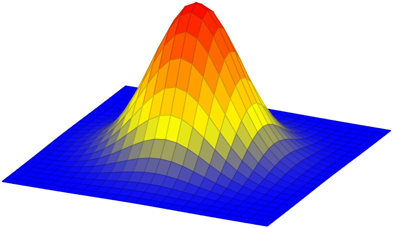


$$f(x, y) = \sqrt{x^2 + y^2}$$

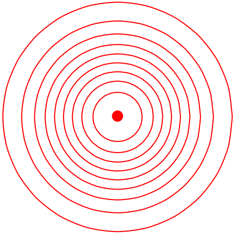
$$\vec{\nabla} f = \frac{\langle x, y \rangle}{\sqrt{x^2 + y^2}}$$

Local minimum

Another Example of Local Maximum

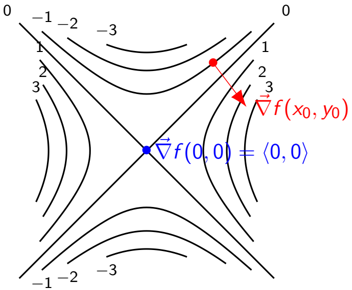


$$z = e^{-x^2-y^2}$$



$$x^2 + y^2 = -\ln(h)$$

Saddle Point



- A point (x_0, y_0) is a saddle point of the graph $z = f(x, y)$, if the contours of f near (x_0, y_0) look like the above
- Examples
  - $(0, 0)$ for the function $f(x, y) = x^2 - y^2$
  - $(0, 0)$ for the function $f(x, y) = xy$

Critical Point of Function

- A critical point of a function f is a point in the domain of f where $\vec{\nabla} f$ is either undefined or equal to the zero vector
- Examples
  - $f(x,y) = \sqrt{x^2+y^2}$: $\vec{\nabla} f(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$ is undefined at $(0,0)$
  - $f(x,y) = e^{-x^2-y^2}$: $\vec{\nabla} f(x,y) = -2e^{-x^2-y^2}\langle x, y, \rangle$ is the zero vector at $(0,0)$
  - $f(x,y) = x^2 - y^2$: $\vec{\nabla} f(x,y) = 2\langle x, -y, \rangle$ is the zero vector at $(0,0)$
- If $\vec{\nabla} f(x_0, y_0) = \langle 0, 0 \rangle$, then the tangent plane

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = f(x_0,y_0)$$

is horizontal

Types of Critical Points

- Suppose (x_0, y_0) is a critical point of a function $f(x, y)$
- A point (x_0, y_0) is a **local maximum**, if

$$f(x, y) \leq f(x_0, y_0) \text{ for all } (x, y) \text{ near } (x_0, y_0)$$

  - Example: $(0, 0)$ for $f(x, y) = -x^2 - y^2$
- A point (x_0, y_0) is a **local minimum**, if

$$f(x, y) \geq f(x_0, y_0) \text{ for all } (x, y) \text{ near } (x_0, y_0)$$

  - Example: $(0, 0)$ for $f(x, y) = x^2 + y^2$
- A point (x_0, y_0) is a **saddle point**, if it meets the criteria in previous slide
  - Example: $(0, 0)$ for $f(x, y) = x^2 - y^2$
- There are other types of critical points that we will not study
  - Example: $(0, 0)$ for $f(x, y) = xy(x^2 - y^2)$

Tests for Critical Point Type

- Analyze formula of function
- Draw graph
- Draw contours
- Second derivative test

Second Derivative Test For Function of One Variable

Suppose x_0 is a critical point of a function $f(x)$, where $f'(x_0) = 0$ and $f''(x_0)$ is defined

- $f''(x_0) > 0 \implies$ local minimum
- $f''(x_0) < 0 \implies$ local maximum
- $f''(x_0) = 0 \implies$ inconclusive

Hessian of Function of Two Variables

- The Hessian of a function $f(x, y)$ at (x_0, y_0) is the matrix

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}$$

- H is a matrix of numbers. There should be no x or y in the formula for H
- The determinant of H is defined to be

$$\det H = H_{11}H_{22} - H_{12}H_{21}$$

Second Derivative Test for Function of Two Variables

- The second derivative test of a function $f(x, y)$ at a critical point (x_0, y_0), where $\vec{\nabla} f(x_0, y_0) = \langle 0, 0 \rangle$ and the Hessian is defined
  - If $\det H(x_0, y_0) = 0$, then the test is inconclusive
    - The shape of the surface near (x_0, y_0) can be simple or complicated
    - Look at contours to learn more
  - If $\det H(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point
  - If $\det H(x_0, y_0) > 0$, then there are two possibilities
    - If $H_{11}(x_0, y_0) > 0$ (or, equivalently, $H_{22}(x_0, y_0) > 0$), then (x_0, y_0) is a local minimum
    - If $H_{11}(x_0, y_0) < 0$ (or, equivalently, $H_{22}(x_0, y_0) < 0$), then (x_0, y_0) is a local maximum

Basic Examples

- $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + c$, where $a, b \neq 0$
  - $\vec{\nabla} f = 2\left\langle a^{-2}x, b^{-2}y \right\rangle$
  - Only one critical point: $(0,0)$
  - $H = 2\begin{bmatrix} a^{-2} & 0 \\ 0 & b^{-2} \end{bmatrix}$
  - $\det H(0,0) = 4a^{-2}b^{-2} > 0$ and $H_{11} = 2a^{-2} > 0$
  - $(0,0)$ **is a local minimum**
- $f(x,y) = -\frac{x^2}{a^2} - \frac{y^2}{b^2} + c$, where $a, b \neq 0$
  - $\vec{\nabla} f = -2\left\langle a^{-2}x, b^{-2}y \right\rangle$
  - Only one critical point: $(0,0)$
  - $H = -2\begin{bmatrix} a^{-2} & 0 \\ 0 & b^{-2} \end{bmatrix}$
  - $\det H(0,0) = 4a^{-2}b^{-2} > 0$ and $H_{11} = -2a^{-2} > 0$
  - $(0,0)$ **is a local maximum**

Basic Examples

- $f(x,y) = -\frac{x^2}{a^2} + \frac{y^2}{b^2} + c$, where $a, b \neq 0$
  - $\vec{\nabla} f = -2\langle a^{-2}x, b^{-2}y\rangle$
  - Only one critical point: $(0,0)$
  - $H = 2\begin{bmatrix} -a^{-2} & 0 \\ 0 & b^{-2} \end{bmatrix}$
  - $\det H(0,0) = -4a^{-2}b^{-2} < 0$
  - $(0,0)$ **is a saddle point**
- $f(x,y) = axy + c$, where $a \neq 0$
  - $\vec{\nabla} f = a\langle y, x\rangle$
  - Only one critical point: $(0,0)$
  - $H = 2\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$
  - $\det H(0,0) = -4a^2 < 0$
  - $(0,0)$ **is a saddle point**

Examples Where Second Derivative Test Fails

- $f(x,y) = x^4 + y^4$
  - $\vec{\nabla} f = 4\langle x^3, y^3\rangle$
  - Only one critical point: $(0,0)$
  - $H = 4\begin{bmatrix} x^3 & 0 \\ 0 & y^3 \end{bmatrix}$
  - $\det H(0,0) = 0$
  - Contours and formula show that $(0,0)$ is a local minimum
- $f(x,y) = (ax+by)^2 + c$, where $ab \neq 0$
  - $\vec{\nabla} f = 2(ax+by)\langle a, b\rangle$
  - All points (x,y), where $ax+by=0$ are critical points
  - $H = 2\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$
  - $\det H(0,0) = a^2b^2 - (ab)^2 = 0$
  - Contours and formula show that surface is a parabolic cylinder and all points in the line $ax+by=0$ are local minimua

Examples Where Second Derivative Test Fails

- Consider the function

$$f(x,y) = xy(x^2 - y^2) = xy(x+y)(x-y)$$

- Its gradient is

$$\begin{aligned}\vec{\nabla} f &= \langle y(x^2 - y^2) + 2x^2 y, x(x^2 - y^2) - 2xy^2 \rangle \\ &= \langle y(3x^2 - y^2), x(x^2 - 3y^2) \rangle \\ &= \langle (y(\sqrt{3}x - y)(\sqrt{3}x + y), x(x - \sqrt{3}y)(x + \sqrt{3}y) \rangle\end{aligned}$$

- Only critical point is $(0,0)$
- $H = \begin{bmatrix} 6xy & 3x^2 - 3y^2 \\ 3x^2 - 3y^2 & -6xy \end{bmatrix}$
- $\det H(0,0) = 0$
- The contour $f = 0$ is given by the equation

$$xy(x+y)(x-y)$$

and consists of the lines $x = 0$, $y = 0$, $y = x$, and $y = -x$

Complicated Example of Second Derivative Test

- $f(x, y) = x^4 + y^4 - 4xy + 1$
  - $\vec{\nabla} f = \langle 4x^3 - 4y, 4y^3 - 4x \rangle$
  - Solve for critical points: $x^3 = y$ and $y^3 = x$
  - Substitute first inequation into second: $x^9 = x$
  - Factor

$$\begin{aligned} 0 &= x^9 - x = x(x^8 - 1) \\ &= x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1) \\ &= x(x - 1)(x + 1)(x^2 + 1)(x^4 + 1) \end{aligned}$$

  - There are three possible values for x: $-1, 0, 1$
  - Since $y = x^3$, the critical points are $(-1, -1)$, $(0, 0)$, $(1, 1)$

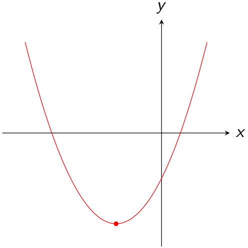
Complicated Example of Second Derivative Test

- $f(x,y) = x^4 + y^4 - 4xy + 1$
- $\vec{\nabla} f = 4\langle x^3 - y, y^3 - x\rangle$
- Critical points are $(-1,-1), (0,0), (1,1)$
- $H = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$
- At the critical points $(-1,1)$ and $(1,1)$
  - $\det H(-1-1) = \det H(1,1) = \det \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix} = 144 - 16 > 0$
  - $H_{11}(-1,-1) = H_{11}(1,1) = 12 > 0$
  - The critical points $(-1,-1)$ and $(1,1)$ are local minima
- At the critical point $(0,0)$
  - $\det H(0,0) = \det \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} = -16$
  - The critical point $(0,0)$ is a saddle point

Global Optimization

- Consider a function $f(x, y)$ on a domain D in 2-space
- A point (x_0, y_0) is a global maximum point, if $f(x, y) \leq f(x_0, y_0)$ for every $(x, y) \in D$. $f(x_0, y_0)$ is the global maximum value.
- There is at most one maximum value but there can be any number, including infinitely many, maximum points
- A point (x_0, y_0) is a global minimum point, if $f(x, y) \leq f(x_0, y_0)$ for every $(x, y) \in D$. $f(x_0, y_0)$ is the global minimum value.
- There is at most one minimum value but there can be any number, including infinitely many, minimum points
- If D has no boundary, then global optimum points are all critical points
- If D has a boundary then global optimum points are either critical points or boundary points

Global Optimization on the Real Line



- Suppose $f(x)$ is a smooth function on the entire real line
- Optimal values, if they exist, must occur at a critical point
- To find optima:
  - Study what happens when $x \to \pm\infty$
  - Find all critical points and calculate f at each of them
- In picture:
  - $f(x) \to +\infty$ as $x \to \pm\infty$, which implies that f has no maximum value
  - f is bounded from below, which means that it has a minimum value
  - There is only one critical point, so that has to be the minimum

Global Optimization in 2-Space

- Find rectangular cardboard box without a top that encloses a given volume V but using the minimum amount of cardboard
- If dimensions of box are H by W by D, then

$$\text{Volume } V = HWD$$
$$\text{Area of card board } A = 2(HW + HD) + WD$$

- V is constant, and we want to minimize A
- Elminiate one variable $H = \frac{V}{WD}$:

$$A(W, D) = 2\frac{V}{WD}(W + D) + WD = 2V(\frac{1}{D} + \frac{1}{W}) + WD$$

Optimal Cardboard Box

- $A(W,D) = 2V(\frac{1}{D} + \frac{1}{W}) + WD$
- Solution must be at a critical point of A
- Find critical points:

$$A_W = -\frac{2V}{W^2} + D = 0, \qquad A_D = -\frac{2V}{D^2} + W = 0$$

$$D = \frac{2V}{W^2}, \qquad W = \frac{2V}{D^2} = 2V\frac{W^4}{4V^2} = \frac{W^4}{2V}$$

- Therefore,

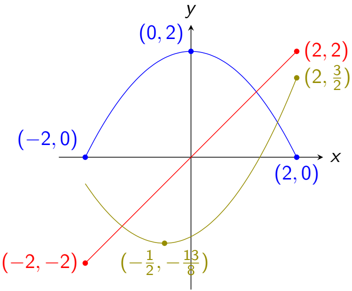
$$0 = \frac{W^4}{2V} - W = W\left(\frac{W^3}{2V} - 1\right)$$

- Since $W \neq 0$,

$$W = (2V)^{1/3} \qquad D = \frac{2V}{W^2} = (2V)^{1/3}$$

$$H = \frac{V}{WD} = \frac{V}{(2V)^{2/3}} = 2(2V)^{1/3}$$

Global Optimization on a Bounded Interval



- The global optima of a smooth function on a bounded closed interval are always at critical or end points
- Here, we have three functions:

$$f(x) = 2 - \frac{1}{2}x^2$$

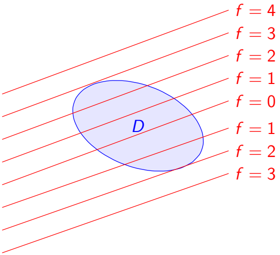
$$g(x) = x$$

$$h(x) = \frac{1}{2}(x^2 - x - 3)$$

Finding Optimal Values and Points on an Interval

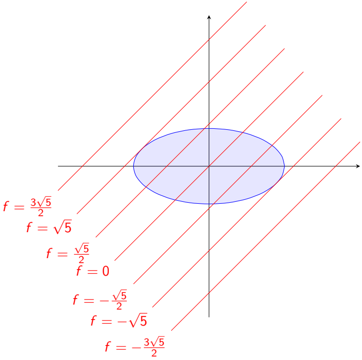
- Find all of the critical points that lie in the interval
- Calculate the value of the function at each critical and each end point
- Identify where the function is maximum and where it is minimum

Global Optima on a Bounded Domain in 2-Space



- Suppose $D = \{(x, y) \ : \ g(x, y) \leq 1\}$
- Maximize or minimize $f(x, y)$ with (x, y) restricted to the domain D
- An optimal point must be either a critical point or a point on the boundary
- If optimal point is on bondary, then it must be at a point where the contour of f and the boundary are tangent
  - Where $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$ for some scalar λ

Example



- Optimize $f(x,y)=y-x$ over all (x,y) such that $\frac{x^2}{4}+y^2\leq 1$
- Since $\vec{\nabla} f=\langle -1,1\rangle$, there are no critical points
- The boundary is the contour $g=1$, where $g(x,y)=\frac{x^2}{4}+y^2$
- Solve for x,y,λ such that

$$\vec{\nabla} f(x,y)=\lambda\vec{\nabla} g(x,y) \text{ and } g(x,y)=1$$